PHYS 225 – Spring 2011 **Marie Lopez del Puerto**

HW #22

1. One way of calculating the population of a city is to find the density of houses in that city (i.e. the number of houses per unit area), multiply that by the probability of finding a given number of humans in a house, and finally, integrate the result over the area of the city. The problem is working out the chances of a house being occupied by a certain number of people, using a mathematical formula. If g(A) is the density of houses, and f(A) is the probability that a house is occupied by a certain number of people,
   - **a.** Write down the integral that would give you the population of a city.
   - **b.** For a hypothetical city, whose area is a circle, f and g are given by:

$$f(r,\theta) = 4e^{-(r-5)^2/20km^2}$$

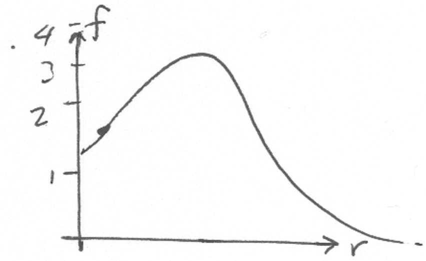
$$g(r,\theta) = 1000e^{-r^2/200km^2}$$

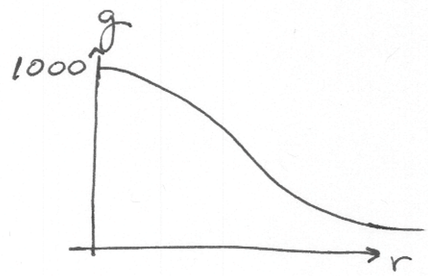
where distances are in units of kilometers, and house density is given in units of houses / km^2.

   - **i.** Sketch f and g as a function of r. What information do they give you?
   - **ii.** If the city limits are at a radius R = 15 km from the city centre, what is the total population of the city? (You might need Mathematica to calculate the integral, or look it up in an integral table).
   - **iii.** What percentage of the population lives downtown? Define downtown as within 1 km of the city centre.
   - **iv.** At what distance R from the city centre is the average number of people per household the highest?
   - **v.** At what distance R from the city centre is the density of houses the highest?
   - **vi.** What is the average household occupation for the entire city?
   - **vii.** Make a 2D (contour or density) plot of g and compare it to the satellite Google map for Houston, TX (if you remove the labels on the map it is easier to see the housing concentration). Is this model realistic? How would you improve it?

a) population $= \int f(A)\, g(A)\, dA$

b) i.





g is the density of houses (number of houses per unit area), it is highest at the center of the city (r=0).

f is the probability that a house is occupied by a certain number of people; the maximum occupancy is 4.

ii. population $= \int f(A)\, g(A)\, dA = \int_0^{2\pi}\int_0^{R} f(r,\theta)\, g(r,\theta)\, r\, dr\, d\theta =$

$$= \int_0^{2\pi}\int_0^{15} 4000\, e^{-(r-5)^2/20}\, e^{-r^2/200}\, r\, dr\, d\theta = 784{,}676$$

iii. % downtown population $= \dfrac{\int_0^{2\pi}\int_0^{1} 4000\, e^{-(r-5)^2/20}\, e^{-r^2/200}\, r\, dr\, d\theta}{784{,}676} \times 100 = 62.6\%$.

iv. we must find the maximum for $f(r,\theta)$:

$$\frac{df}{dr} = 4 e^{-(r-5)^2/20} \cdot \left(-\frac{2(r-5)}{20}\right) = 0$$

$$-8\, e^{-(r-5)^2/20}\, \frac{(r-5)}{20} = 0 \Rightarrow r = 5 \text{ km}$$

v. we must find the maximum for $g(r,\theta)$:

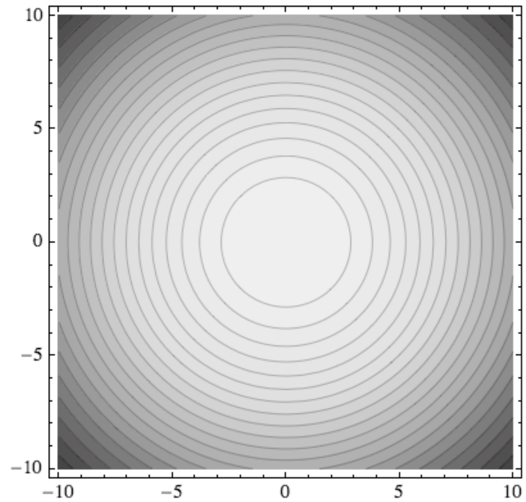
$$\frac{dg}{dr} = 1000\, e^{-r^2/200}\left(-\frac{2r}{200}\right) = 0 \Rightarrow r = 0$$

vi. average household occupation $= \dfrac{\text{population}}{\text{total households}} = \dfrac{784{,}676}{\int_0^{2\pi}\int_0^{15} g(r,\theta)\, r\, dr\, d\theta} =$

$$= \frac{784{,}676}{\int_0^{2\pi}\int_0^{15} 1000\, e^{-r^2/200}\, r\, dr\, d\theta} = 1.85 \text{ people/house.}$$

vii. cities are not usually uniform in θ, the building density is higher along major roads. A better model would have g as a function of θ be a maximum at certain angles, for Houston they would be ~ 0°, 135°, 180°, 225°, 270° + 315°.

Contour plot of g (distances in km):



Satellite image of Houston, TX: