Name: key.

Tutorial: Quantum Tunneling[1]

Some useful equations:

1D Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}+V(x,t)\Psi(x,t)=i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

1D time-independent Sch. eqn:

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2}+V(x)\varphi(x)=E\varphi(x)$$

Solution to Schrödinger equation with $E > V$, when $V = constant$:

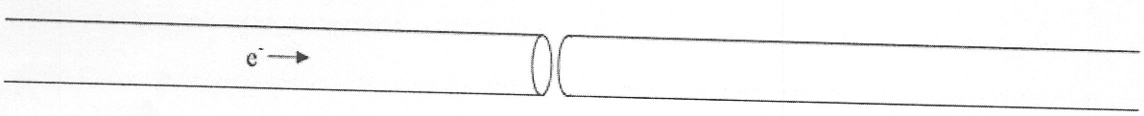
$$\Psi(x,t)=(Ae^{ikx}+Be^{-ikx})e^{-iEt/\hbar}$$

Solution to Schrödinger equation with $E < V$, when $V = constant$:

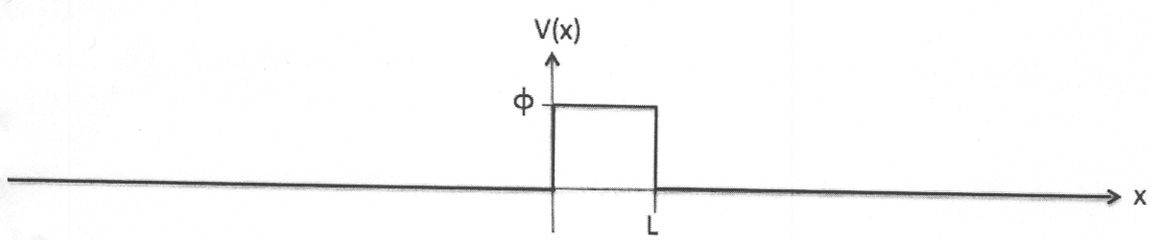
$$\Psi(x,t)=(Ae^{\alpha x}+Be^{-\alpha x})e^{-iEt/\hbar}$$

In this tutorial you will explore the physics of an electron traveling through an air gap in a wire – first in the case in which the electron has enough energy to get through the gap classically, and then in the case in which it does not. If you're paying attention, you should be surprised by some of the results.

Consider an electron initially moving to the right through a very long smooth copper wire with a small air gap in the middle (as shown in the figure below, the wire extends infinitely in the –x and +x directions).



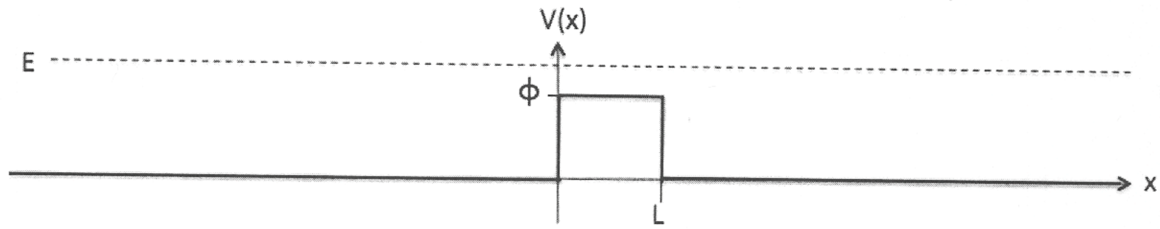
If the air gap is length L, the work function of copper is ϕ and we define V(x) = 0 inside the wire, then the potential energy as a function of space is:



[1] Developed by Sam McKagan, Kathy Perkins, and Carl Wieman, University of Colorado. Modified by Marie Lopez del Puerto, University of St. Thomas.

PART 1: $E > \phi$

Suppose the electron traveling through the wire has an initial energy $E > \phi$, shown by the dashed line:



1. For the region in the copper wire to the left of the air gap, write down the general solution for $\psi(x,t)$. Plug it into the Schrödinger Equation to make sure it works, and show that the total energy E of the electron is given by $E = \hbar^2k^2/(2m)$.

$$\psi(x,t) = \left(Ae^{ikx} + Be^{-ikx}\right)e^{-iEt/\hbar}$$

the Schrödinger eqn. with $V=0$,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t}$$

gives

$$-\frac{\hbar^2}{2m}\left(-k^2\psi\right) = i\hbar\left(-\frac{iE}{\hbar}\psi\right)$$

which simplifies to

$$E = \frac{\hbar^2k^2}{2m}$$

2. For the region in the air gap ($0<x<L$), write down the general solution for $\psi(x,t)$. Note that the value of k can be different than in question 1, so call it k'. Plug your solution into the Schrödinger equation (with $V(x) = \phi$) to make sure it works and show that the total energy E of the electron is given by $E = \hbar^2 k'^2/(2m)+\phi$.

$$\psi(x,t) = \left(Ae^{ik'x} + Be^{-ik'x}\right)e^{-iEt/\hbar}$$

the Schrödinger eqn. with $V=\phi$,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \phi\psi = i\hbar\frac{\partial\psi}{\partial t}$$

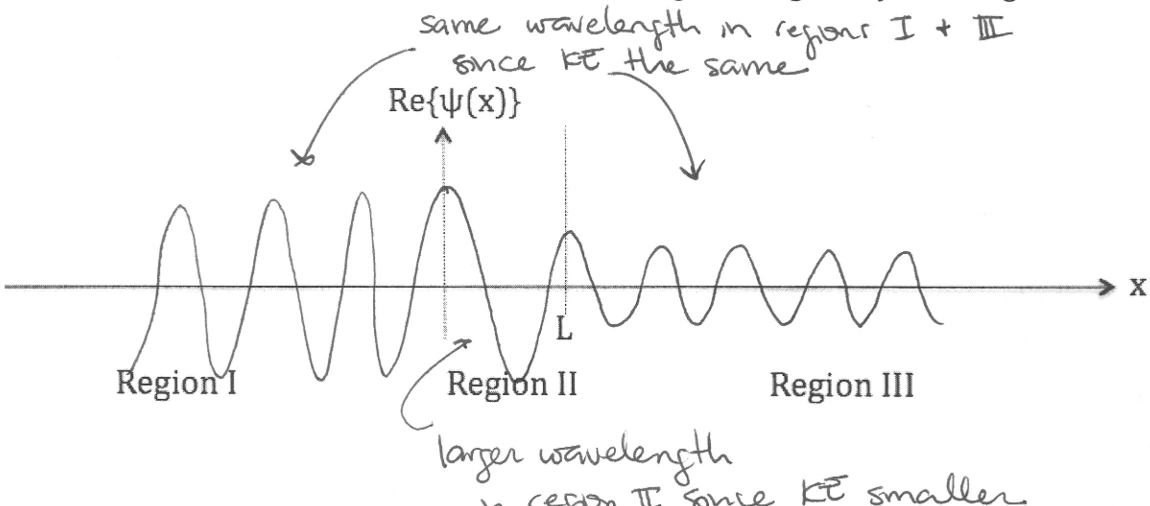
gives $$-\frac{\hbar^2}{2m}\left(-k'^2\psi\right) + \phi\psi = i\hbar\left(-\frac{iE}{\hbar}\psi\right)$$

which simplifies to $$E = \frac{\hbar^2 k'^2}{2m} + \phi$$

3. For the region in the copper wire to the right of the air gap, write down the general solution for $\psi(x,t)$. Is the value of k here the same as either of the values of k above? Why or why not? Plug your solution into the Schrödinger equation to make sure it works. Show that the total energy E of the electron is given by $E = \hbar^2 k^2/(2m)$.

Since here $V=0$ as in question 1, the solution is the same and k is the same, thus the energy is also the same: $$E = \frac{\hbar^2 k^2}{2m}$$

4. In the plot below, sketch the shape of the real part of the wave function at $t = 0$ in each region for the case where $E > \phi$. Don't worry about the relative magnitudes of the waves in the different regions, but think carefully about the general shape of the graph in each region. Remember that the wavelength changes as you change k!



5. What is the basic shape of the real part of the wave function in each of the three regions? For example, is it linear, constant, quadratic, exponential, sinusoidal, or something else?

sinusoidal in all three regions

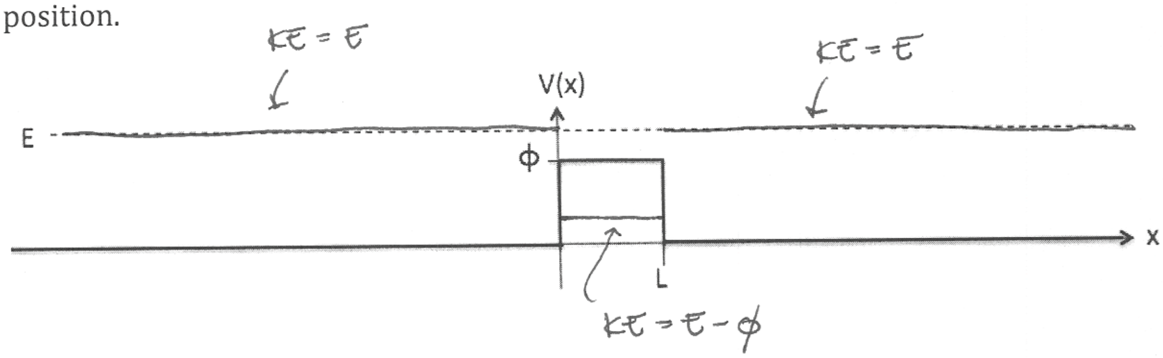
6. Is the total energy of the electron to the right of the air gap *greater than, less than* or *equal to* the total energy of the electron to the left of the air gap? Explain your reasoning.

equal to, energy must be conserved!

7. Fill in the values for the potential, kinetic, and total energy of the particle in each of the three regions in the table below. Your answers should be in terms of E and ϕ.

	Region I (Left wire)	Region II (Air gap)	Region III (Right wire)
Potential energy	0	ϕ	0
Kinetic energy	E	$E-\phi$	E
Total energy	E	E	E

8. In the figure below, sketch the kinetic energy of the electron as a function of position.



9. Write an equation that relates the kinetic energy of a particle to its deBroglie wavelength, i.e. how are *KE* and λ related?

$$KE = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2}$$

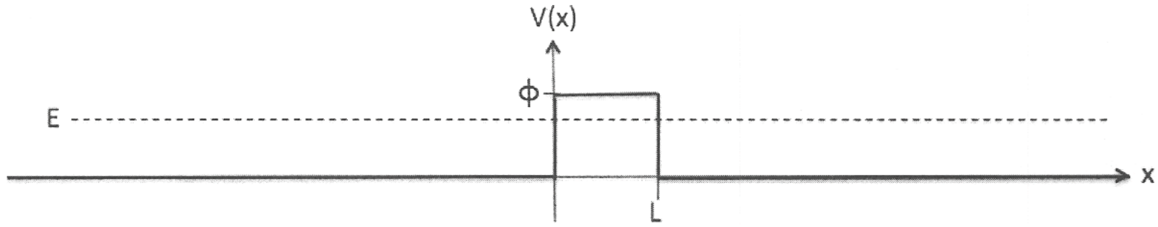
10. Are the wave functions you sketched in question 4 consistent with your equation in question 9 and your kinetic energies in question 7? Resolve any discrepancies.

As KE increases, λ must decrease, and vice versa.

The KE in regions I and III is the same, so the wavelength must also be the same in these regions. The KE is smaller in the airgap (region II), so the wavelength must be longer.

Part II: E < ϕ

Suppose the electron traveling through the wire has an initial energy ***E* < *ϕ***, shown by the dashed line:



11. For the region in the copper wire to the left of the air gap, write down the general solution for *ψ(x,t)*. Plug it into the Schrödinger Equation to make sure it works, and show that the total energy *E* of the electron is given by $E = \hbar^2k^2/(2m)$.

V = 0 like in question 1, so the solution must be the same:

$$\psi(x,t) = (Ae^{ikx} + Be^{-ikx})e^{-iEt/\hbar}$$

and $E = \dfrac{\hbar^2k^2}{2m}$

12. For the region in the air gap ($0<x<L$), write down the general solution for $\psi(x,t)$. Remember that $E < \phi$. Plug your solution into the Schrödinger equation (with $V(x) = \phi$) to make sure it works and show that the total energy E of the electron is given by $E = -\hbar^2\alpha^2/(2m)+\phi$.

$$\psi(x,t) = (Ae^{\alpha x} + Be^{-\alpha x})e^{-iEt/\hbar}$$

Note that growing exponential makes no physical sense (increasing probability of finding the electron in the air gap?) so $A = 0$.

the Schrödinger equation with $V = \phi$,

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \phi\psi = i\hbar\frac{\partial\psi}{\partial t}$$

gives

$$-\frac{\hbar^2}{2m}(\alpha^2\psi) + \phi\psi = i\hbar\left(-\frac{iE}{\hbar}\psi\right)$$

which simplifies to

$$E = -\frac{\hbar^2\alpha^2}{2m} + \phi$$

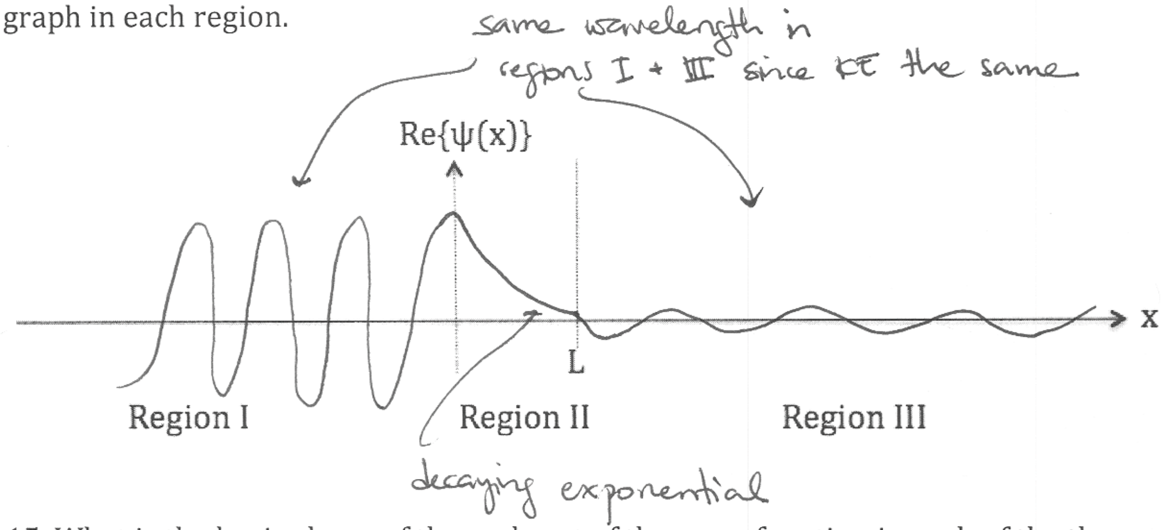
13. For the region in the copper wire to the right of the air gap, write down the general solution for $\psi(x,t)$. Plug your solution into the Schrödinger equation to make sure it works. Show that the total energy E of the electron is given by $E = \hbar^2k^2/(2m)$.

$V = 0$ like in question 1, so the solution must be the same:

$$\psi(x,t) = (Ae^{ikx} + Be^{-ikx})e^{-iEt/\hbar}$$

$$E = \frac{\hbar^2k^2}{2m}$$

14. In the plot below, sketch the shape of the real part of the wave function at $t = 0$ in each region for the case where $E < \phi$. Don't worry about the relative magnitudes of the waves in the different regions, but think carefully about the general shape of the graph in each region.



15. What is the basic shape of the real part of the wave function in each of the three regions? For example, is it linear, constant, quadratic, exponential, sinusoidal, or something else?

sinusoidal in regions I + III, decaying exponential in region II

Although classically the electron does NOT have enough energy to get out of the metal and into the air gap, you should see from your plot above that there is a non-zero probability of finding the electron in the air gap. The probability decays exponentially as you go farther into the gap, but if the gap is thin enough, the electron can "tunnel" through the gap into the copper wire on the right. This is what is meant when it is said that a particle can quantum mechanically tunnel through a barrier.

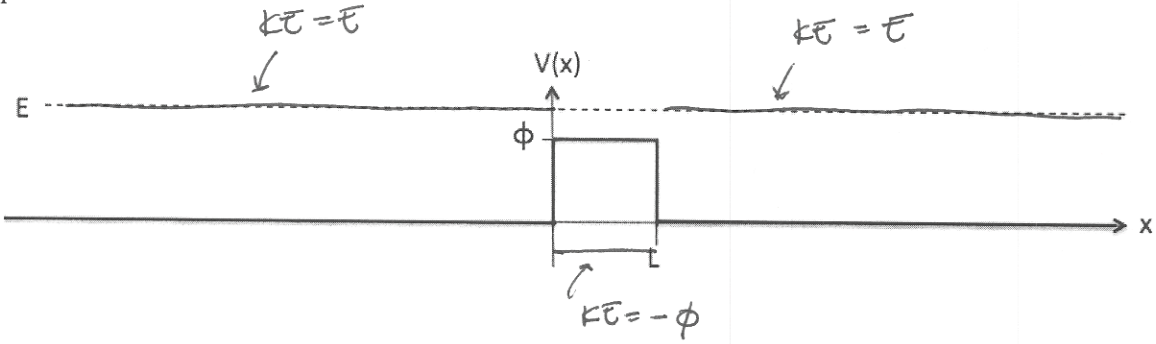
16. Consider an electron that has tunneled through the barrier. Is the total energy of the electron to the right of the air gap *greater than, less than* or *equal to* the total energy of the electron to the left of the air gap? Explain your reasoning.

equal to, the probability of finding the electron in region III is small, but energy must be conserved!

17. Fill in the values for the potential, kinetic, and total energy of the particle in each of the three regions in the table below. Your answers should be in terms of E and ϕ.

	Region I (Left wire)	Region II (Air gap)	Region III (Right wire)
Potential energy	0	ϕ	0
Kinetic energy	E	$E - \phi$	E
Total energy	E	E	E

18. In the figure below, sketch the kinetic energy of the electron as a function of position.



19. Do you notice anything unusual about the kinetic energy?

The kinetic energy is negative in the air gap! The particle doesn't have enough total energy to be in this region but it is 'borrowing' some KE for a short time (such that $\Delta E \Delta t \geq \hbar$) to compensate. There is no classical analogue, but we can describe the behavior of ψ in this region and it accurately predicts the results of experiments.

20. Are the wave functions you sketched in question 14 consistent with your equation in question 9 and your kinetic energies in question 17? Resolve any discrepancies.

KE is the same in regions I + III, so the wavelengths must be the same. In region II the wave function is a decaying exponential.