Oscillation – Set 2

1) In figure (a) below, the position versus time of a SHO, estimate the value of the phase angle, ϕ.

$\pi/2 < \phi < 3\pi/2$ ← in this range

Consider figure (b) below, the velocity versus time of a SHO.

2) Is the particle stationary, moving towards -x, or moving towards +x when the particle is at:

Point A: towards $-x_{max}$

Point B: towards $+x_{max}$

$x(t) = A\cos(\omega t + \phi)$

$v(t) = -A\omega \sin(\omega t + \phi)$

3) Is x=0, x>0, or x<0 when the particle is at:

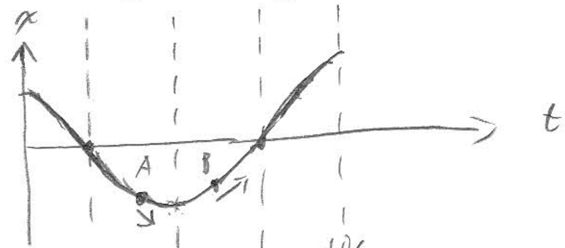
Point A: $x < 0$

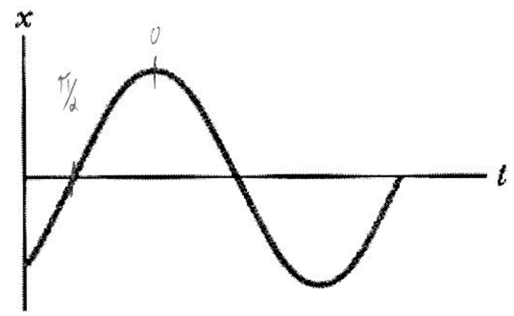
Point B: $x < 0$

4) Is the particle's speed (the magnitude of it's velocity) increasing, decreasing, or constant when the particle is at:

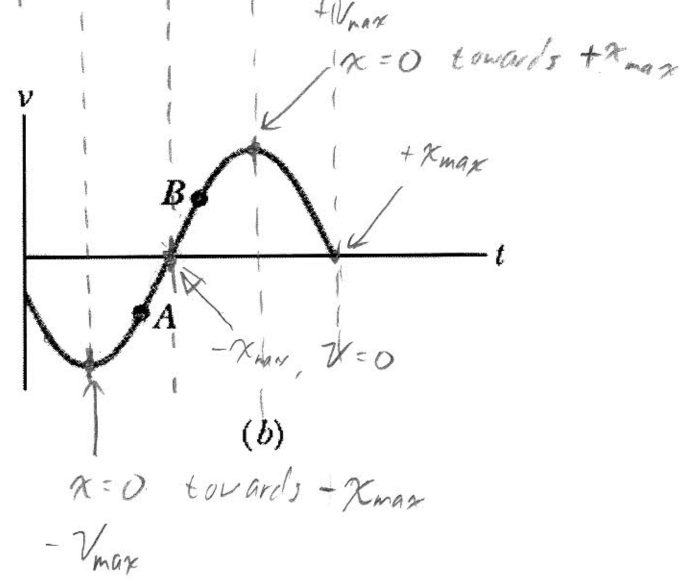
Point A: Decreasing

Point B: increasing





(a)



(b)

Oscillation – Set 2

The acceleration of a particle in Simple Harmonic Motion is plotted in the figure below.

1) Which point(s) represent the particle's acceleration when it is at $x = -x_{max}$?

Point 2

2) Which point(s) represent the particle's acceleration when it is at $x = +x_{max}$?

Point 6

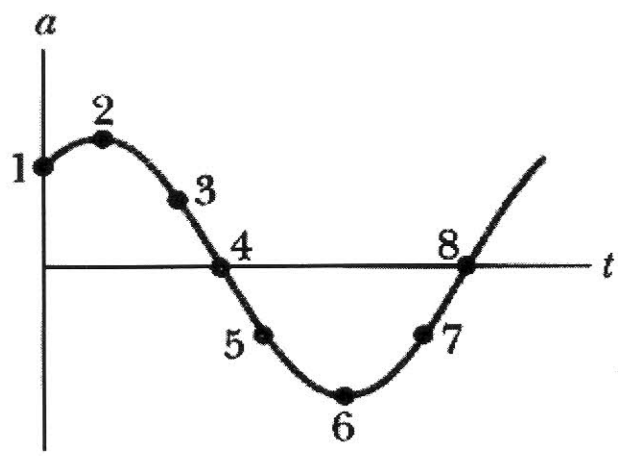
2) At point 4, is the velocity of the particle positive, negative, or zero?

Positive (just after a period of positive accel)

3) At point 5, what is the particle's position?

A) $x = 0$
B) $x = -x_{max}$
C) $x = +x_{max}$
D) $\boxed{0 < x < +x_{max}}$
E) $-x_{max} < x < 0$

$x(t) = A\cos(\omega t + \phi)$
$v(t) = -\omega A \sin(\omega t + \phi)$
$a(t) = -\omega^2 A\cos(\omega t + \phi)$



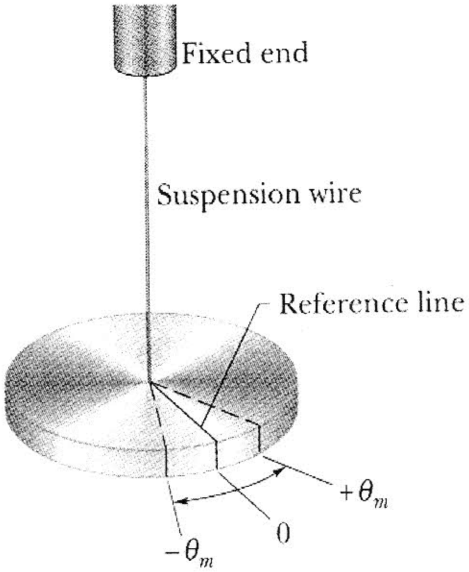
Oscillation – Set 2

The device in the picture below is known as a torsion pendulum. It is a flat disk attached to a length of stiff wire. When the wire is twisted, it responds by providing a torque on the disk, much the same way a spring provides a force when it is stretched. The torque provided by the wire is $T=-\kappa\theta$, where κ (greek letter kappa) is the torsion constant and theta is the angular displacement from equilibrium.

a) The moment of Inertia of the disk is $I=\frac{1}{2}MR^2$. Using the rotational version of Newton's Second Law, find the oscillator frequency of the torsion pendulum.

b) If a solid bar of length L, $I=\frac{1}{12}ML^2$, were suspended from the wire, what would the oscillator frequency be?

Let's solve the problem generally in terms of I, then plug in the different I for each object.



NSL for rotation

$$\Sigma T = I\alpha$$

$$-k\theta = I\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \boxed{\frac{d^2\theta}{dt^2} = -\frac{k}{I}\theta}$$ SHO!

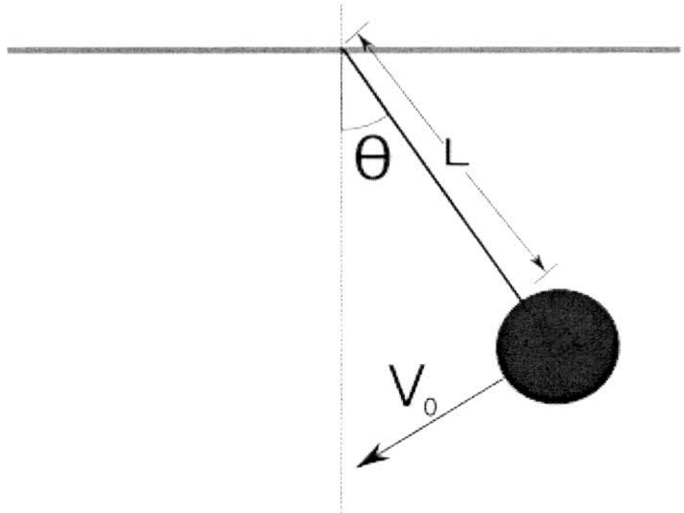
This is the simple harmonic oscillator equation, except in θ instead of x.

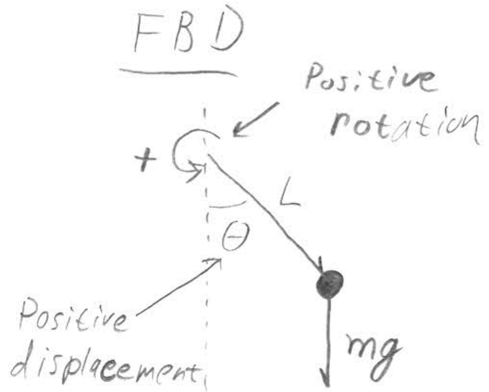
$$\boxed{\omega = \left(\frac{k}{I}\right)^{1/2}} \qquad a)\ \omega = \left[\frac{2k}{MR^2}\right]^{1/2} \qquad b)\ \omega = \left[\frac{12k}{ML^2}\right]^{1/2}$$

Below is a simple pendulum consisting of a massless rod of length L with a point mass if mass m attached to the end.

a) Find the frequency of small oscillations of the pendulum.

b) At t=0, the pendulum makes an angle θ_0 with the vertical and the point mass has a velocity V_0. What is ~~the amplitude of the oscillator?~~ Phase angle?





$\vec{L} \times m\vec{g} \Rightarrow \otimes$ gives negative torque by rhr

NSL

$\sum \tau = I\alpha, \quad I = mL^2$

$-\cancel{m}g\cancel{L}\sin\theta = \cancel{m}L^{\cancel{2}}\dfrac{d^2\theta}{dt^2}$

$\Rightarrow \dfrac{d^2\theta}{dt^2} = -\dfrac{g}{L}\sin\theta$, Almost a SHO.

For small θ, $\sin\theta \cong \theta$ (small angle approximation)

So; For small oscillations: $\boxed{\dfrac{d^2\theta}{dt^2} = -\dfrac{g}{L}\theta}$ SHO!

$\boxed{\omega = \sqrt{\dfrac{g}{L}}}$

continued ↓

Oscillation Set 2, P2 continued

b) $\theta(0) = \theta_0$, $v(0) = v_0 = \omega_0 L$

$\boxed{\text{!!Be very careful!!}}$

This angular velocity is not the same ω as the oscillator frequency, $\sqrt{g/L}$. Let's call $\boxed{\sqrt{g/L} = \omega_F}$

Angular versions of SHO general solution

$$\theta(t) = A\cos(\omega_F t + \phi)$$
$$\omega(t) = -\omega_F A \sin(\omega_F t + \phi)$$

$$\theta_0 = A\cos(\phi)$$
$$\frac{v_0}{L} = -\omega_F A \sin(\phi)$$
$$\Rightarrow \frac{v_0}{L\theta_0} = \frac{-\omega_F \cancel{A} \sin(\phi)}{\cancel{A}\cos(\phi)}$$

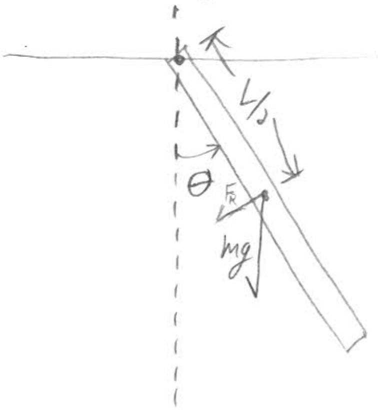
$$\Rightarrow \boxed{\tan(\phi) = -\frac{v_0}{\omega_F L \theta_0}}$$

$$\Rightarrow \tan(\phi) = -\left(\frac{L}{g}\right)^{1/2} \frac{v_0}{L\theta_0}$$ plug in ω_F

$$\Rightarrow \boxed{\tan(\phi) = -\frac{v_0}{\theta_0 \sqrt{gL}}}$$

A meter stick with a mass M is suspended from one end and allowed to swing like a pendulum.

a) What is its **period** of small oscillations?
b) What length *L* does a simple pendulum (a point mass attached to a massless rod) need in order to have the same period?



$I = \frac{1}{3}ML^2$, $F = -mg\sin\theta$, $L = 1$ meter

$\sum T = I\alpha$

$-(mg\sin\theta)\frac{L}{2} = I\alpha$

$-mgL\sin\theta = 2I\alpha$

$-\cancel{m}g\cancel{L}\sin\theta = 2\cdot\frac{1}{3}\cancel{m}L^{\cancel{2}}\frac{d^2\theta}{dt^2}$

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{3}{2}\frac{g}{L}\sin\theta}$$

a) For small oscillations, $\sin\theta \cong \theta$

$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{3}{2}\frac{g}{L}\theta \Rightarrow \omega = \left(\frac{3}{2}\frac{g}{L}\right)^{1/2}$

and $T = \frac{2\pi}{\omega}$, $L = 1$ m, $T = 2\pi\left(\frac{2}{3g}\right)^{1/2} = \boxed{1.6\ s}$

b) From problem 2, the frequency of a simple pendulum is

$\omega = \sqrt{g/L_s}$ so $T = \frac{2\pi}{\omega} \Rightarrow T = 2\pi\sqrt{L_s/g}$

We want $T_m = T_s$ so: $\cancel{2\pi}\sqrt{L_s/g} = \cancel{2\pi}\left(\frac{2L}{3g}\right) \Rightarrow \frac{L_s}{\cancel{g}} = \frac{2}{3}\frac{1}{\cancel{g}}$

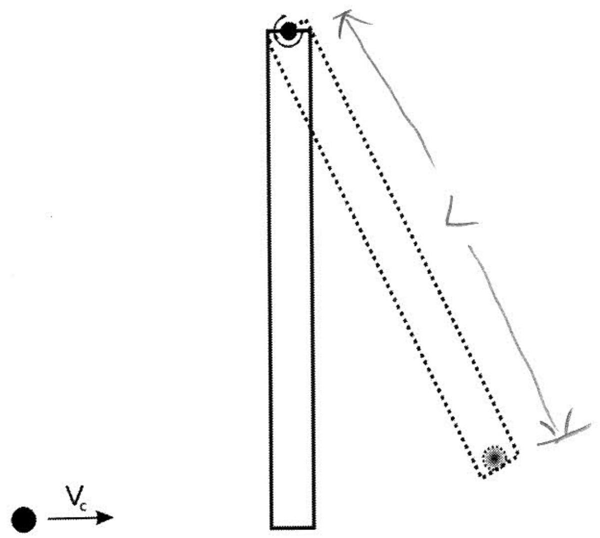
meter stick ↑ simple ↑

$$\boxed{L_s = \frac{2}{3}L}$$

Oscillation – Set 2

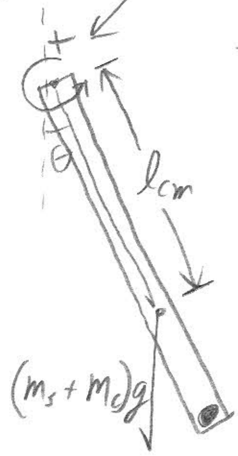
A 1 kg meter stick is hung from its and allowed to pivot. A small wad of clay with a mass of 0.25 kg with a velocity $V_c = 2$ m/s impacts the bottom of the meter stick. Assuming that the resulting oscillations are small:

a) Find the angular frequency of the resulting pendulum.
b) Find the phase angle of the resulting oscillator.
~~c) Find the amplitude of the oscillations.~~



Positive displacement

Then, $\vec{r}\times\vec{F}$ is neg.



so: $-(m_s+m_c)g\,l_{cm}\sin\theta = I\alpha$

$$\Rightarrow \boxed{\alpha = \frac{-(m_s+m_c)l_{cm}g}{I}\sin\theta} \quad (1)$$

But, for small θ, $\boxed{\sin\theta \cong \theta}$; (2)

$$I = I_s + I_c \Rightarrow \boxed{I = \tfrac{1}{3}m_sL^2 + m_cL^2} \quad (3)$$

$$\boxed{l_{cm} = \frac{m_s\frac{L}{2} + m_cL}{m_s + m_c}} \quad (4)$$

$$\Rightarrow \alpha = \frac{-\cancel{(m_s+m_c)}\,g}{(\frac{1}{3}m_s+m_c)L^{\cancel{2}}}\cdot\frac{(\frac{1}{2}m_s+m_c)\cancel{L}}{\cancel{(m_s+m_c)}}\,\theta$$

$$\Rightarrow \alpha = -\boxed{\underbrace{\frac{\frac{1}{2}m_s+m_c}{\frac{1}{3}m_s+m_c}\,\frac{g}{L}}_{\omega^2}}\,\theta \qquad \boxed{\omega = \left[\frac{\frac{1}{2}m_s+m_c}{\frac{1}{3}m_s+m_c}\,\frac{g}{L}\right]^{1/2}}$$

continued ↓

Oscillation Set 2, P6 - continued

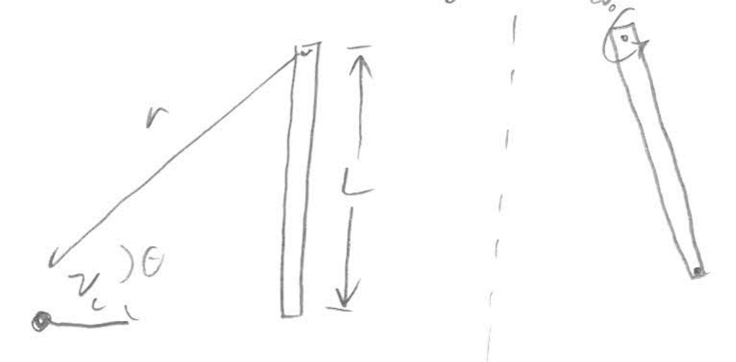
b) $\theta(t) = \theta_{max}\cos(\omega t + \phi)$

$\omega(t) = -\omega\,\theta_{max}\sin(\omega t + \phi)$

NOT the oscillator frequency!

collision at $t=0$: $\theta(0) = 0$

Conserve angular momentum to find $\omega(0) = \boxed{W_0}$



$r m v_c \sin\theta = I W_0$

$(\cancel{m_s}+m_c) v_c \cancel{L} = (\tfrac{1}{3}m_s + m_c)L^{\cancel{2}} W_0 \Rightarrow \boxed{W_0 = \frac{\cancel{m_s}+m_c}{\frac{1}{3}m_s + m_c}\frac{v_c}{L}}$

so W_0 is pos.

Find phase angle:

$\theta_0 = \theta_{max}\cos(\cancel{\omega t}^{0} + \phi) \Rightarrow 0 = \cos(\phi) \Rightarrow \boxed{\phi = \pi/2,\ 3\pi/2}$

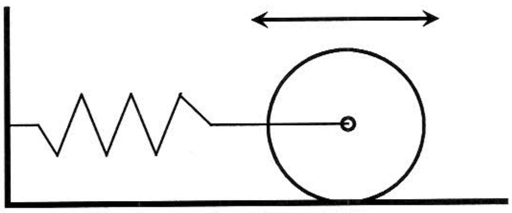
c) $W_0 = -\omega\,\theta_{max}\sin(\phi)$, $\phi = 3\pi/2$ so $\sin(\phi) = -1$

$\Rightarrow W_0 = +\omega\,\theta_{max} \Rightarrow \theta_{max} = \dfrac{W_0}{\omega} \Rightarrow \theta_{max} = \dfrac{m_s + m_c}{\frac{1}{3}m_s + m_c}\dfrac{v_c}{L}\left[\dfrac{\frac{1}{3}m_s + m_c}{\frac{1}{2}m_s + m_c}\dfrac{L}{g}\right]^{1/2}$

$$\boxed{\theta_{max} = (m_s + m_c)\, v_c/g\, \frac{1}{\left[(\frac{1}{3}m_s + m_c)(\frac{1}{2}m_s + m_c)L\right]^{1/2}}}$$

Oscillation – Set 2

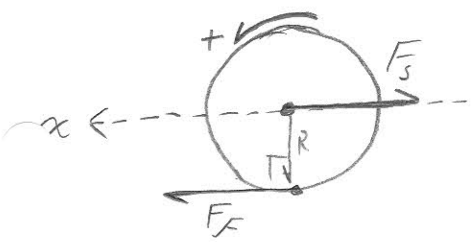
A solid cylinder of mass M is attached to a horizontal spring with force constant k. The cylinder can roll without slipping along the horizontal plane. When the system is displaced from the equilibrium position, it executes simple harmonic motion. Derive an expression for the period of the oscillations in terms of M, k, I and R.



This oscillator is rotating and translating, so we need Torque and translation.

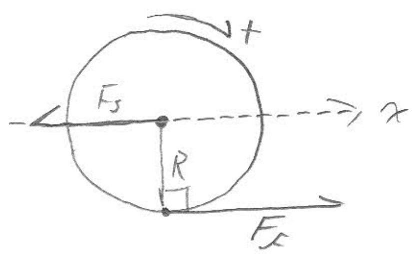
FBD → Pick the initial displacement from equilibrium as positive translation and rotation.

Displace left



Just to show that we can do it either way...

Displace right



① $\sum \tau = I\alpha \Rightarrow \boxed{-F_f R = I\alpha}$ ← same → $\sum \tau = I\alpha \Rightarrow \boxed{-F_f R = I\alpha}$

② $\sum F = ma \Rightarrow \boxed{F_f - F_s = ma}$ ← same → $\sum F = ma \Rightarrow \boxed{F_f - F_s = ma}$

solve ① for F_f and subst. into ②:

spring

replaced α

$$F_f = -\frac{I}{R}\alpha \xrightarrow{\text{into ②}} -\frac{I}{R}\alpha - kx = ma \Rightarrow -kx = ma + \frac{I}{R^2}a$$

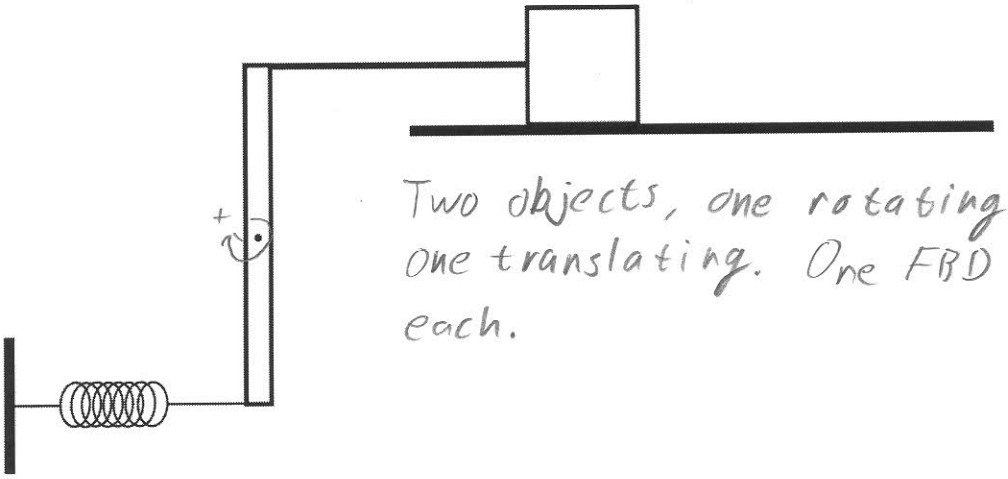
ω^2

$$\Rightarrow \frac{d^2x}{dt^2} = -\boxed{\frac{k}{I/R^2 + m}}\,x$$

Period

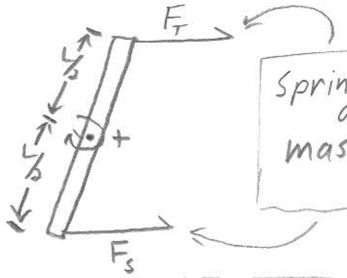
$$T = \frac{2\pi}{\omega} \Rightarrow \boxed{T = 2\pi\left[\frac{I/R^2 + m}{k}\right]^{1/2}}$$

A block of mass M resting on a frictionless surface is attached to a stiff rod of negligible mass. The other end of the rod is attached to the top of a thin bar of length l mass M that is allowed to rotate about its center. The bottom of a bar is attached to a light spring of spring constant k. The spring is relaxed when the bar is vertical. Find the frequency of small oscillations.



Two objects, one rotating and one translating. One FBD for each.

FBD – Displace rod clockwise.



Spring spins rod ccw
mass resists ccw motion

other half of F_T force pair

$$\Sigma T = I\alpha \Rightarrow \boxed{-\frac{L}{2}F_s + \frac{L}{2}F_T = I\alpha} \quad ①$$

$$\boxed{-F_T = ma} \quad ②$$

Subst: ② → ① $-\frac{L}{2}kx - \frac{L}{2}ma = I\alpha$

hmm... rot or tran. Let's go tran...

$$\frac{L}{2}\alpha = a \Rightarrow \alpha = \frac{2a}{L}, \quad I = \frac{1}{12}mL^2$$

$$\Rightarrow -\frac{\cancel{L}}{\cancel{2}}kx = \frac{\cancel{L}}{\cancel{2}}ma + \frac{1}{\cancel{12}_{6}}m\cancel{L^2}\frac{2a}{\cancel{L}} \Rightarrow -kx = ma + \frac{1}{3}ma$$

$$\Rightarrow -kx = \frac{4}{3}ma \Rightarrow \frac{d^2x}{dt^2} = -\underbrace{\boxed{\frac{3}{4}\frac{k}{m}}}_{\omega^2}x$$