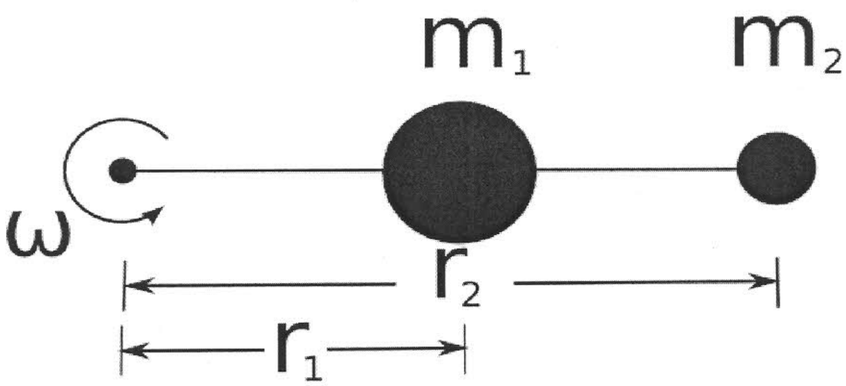
Consider a thin (essentially massless) bar with two masses attached to it as pictured below. The bar is rotating about the point shown in the diagram with an angular velocity ω.



a) Write an expression for the total kinetic energy of the system in terms of r_1, r_2, and ω. Simplify your expression as much as possible.

$$K_T = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2$$

$$= \tfrac{1}{2}m_1 r_1^2\omega^2 + \tfrac{1}{2}m_2 r_2^2\omega^2$$

$$\boxed{K_T = \tfrac{1}{2}(m_1 r_1^2 + m_2 r_2^2)\omega^2}$$

b) Generalize the expression above to a system with *n* masses (use a summation symbol, Σ, in your expression).

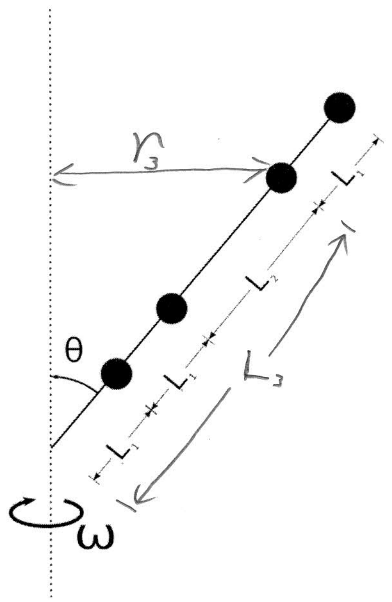
$$\boxed{K_T = \tfrac{1}{2}\left(\sum m_i r_i^2\right)\omega^2}$$

The term in parenthesis is the moment of inertia.

$I = \sum m_i r_i^2$, for discrete masses anyway...

Four point masses, each of mass m, are attached to a rigid massless rod that makes an angle θ with the axis of rotation. Let $L_2 = 2L_1$.

a) What is the moment of inertia of this system?
b) What is the kinetic energy of this system if it's rotating with angular velocity ω.



$r_3 = L_3 \sin\theta$

$\underline{r_3 = 4L_1 \sin\theta}$

$$I = \sum m_i r_i^2$$

$$= mr_1^2 + mr_2^2 + mr_3^2 + mr_4^2$$

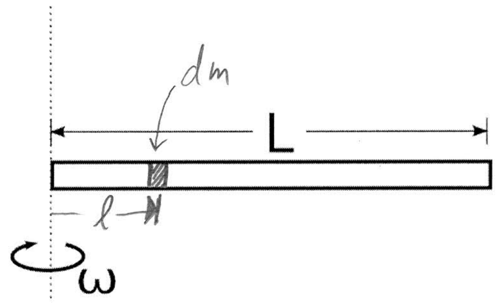
$$= m(r_1^2 + r_2^2 + r_3^2 + r_4^2)$$

$$= m\left(L_1^2 \sin^2\theta + (2^2)L_1^2 \sin^2\theta + (4^2)L_1^2 \sin^2\theta + (5^2)L_1^2 \sin^2\theta\right)$$

$$= mL_1^2 \sin^2\theta\,(1 + 4 + 16 + 25)$$

$$\boxed{I = 46 mL_1^2}$$

Calculate the moment of inertia of a uniform bar of length L and mass M about the axis of rotation shown.



$$\lambda = \frac{M}{L}$$

$$I = \int r^2 dm, \qquad dm = \lambda\, dl$$

$$dm = \frac{M}{L} dl, \quad \underline{r = l}$$

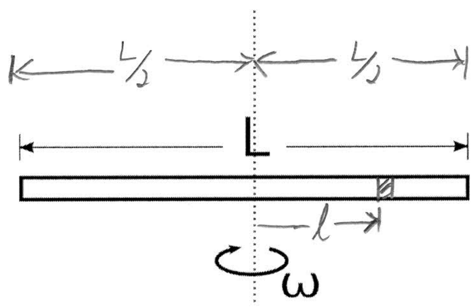
$$I = \int_0^L l \frac{M}{L} dl = \frac{M}{L}\int_0^L l\, dl$$

$$= \frac{M}{L}\left(\tfrac{1}{3} l^3\right)\Big|_0^L$$

$$= \frac{M}{\cancel{L}} \tfrac{1}{3} L^{\cancel{3}\,2}$$

$$\boxed{I = \tfrac{1}{3} M L^2}$$

Calculate the moment of inertia of a uniform bar of length L and mass M about the axis of rotation shown.



$\lambda = \frac{M}{L}$

$$I = \int r^2\,dm, \qquad dm = \lambda\,dl$$

$$dm = \frac{M}{L}\,dl, \quad \underline{r = l}$$

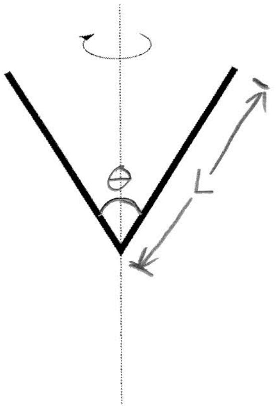
$$I = \int_{-L/2}^{L/2} l^2 \frac{M}{L}\,dl = \frac{M}{L}\int_{-L/2}^{L/2} l^2\,dl = \frac{M}{L}\left(\tfrac{1}{3} l^3\right)\Big|_{-L/2}^{L/2}$$

$$= \frac{M}{L}\frac{1}{3}\left(\frac{L^3}{8} + \frac{L^3}{8}\right) = \frac{M}{\cancel{L}}\frac{1}{3}\frac{L^{\cancel{3}\,2}}{4}$$

$$\boxed{I = \frac{1}{12} M L^2}$$

Rotation – Set 2

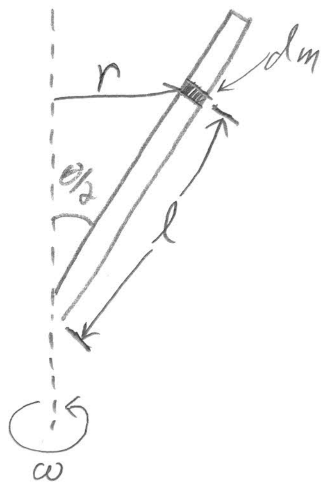
Calculate the moment of inertia of the bent rod of mass **M** shown in the figure below. The rotation axis is in the plane of the "V" bisecting it at the vertex. The rod is bent at an angle $\boldsymbol{\theta}$ and each leg has a length **L**.



$$\lambda = \frac{M}{2L}$$

Each leg has Length L and there are two legs.

The system is symmetric. For every dm on the right, there is on an equal distance from the axis on the left.
So, we can calculate just one leg and multiply by 2.



$$I = \int r^2 dm$$

$r \Rightarrow$ distance from mass to axis of rot.

$dm \Rightarrow$ mass element of the rod

$l \Rightarrow$ is the variable of integration since we are summing pieces of dm along the rod

So: $dm = \lambda dl \Rightarrow dm = \frac{M}{2L} dl$

$$\boxed{I = 2\int_0^L r^2 \frac{M}{2L} dl}$$

Continued ↓

Since we're integrating in l, we must write r in terms of l.

Looking at the digram: $r = l\sin\left(\frac{\theta}{2}\right)$

So: $$I = \cancel{2}\int_0^L l^2 \sin^2\left(\frac{\theta}{2}\right) \frac{M}{\cancel{2}L}\,dl$$

$$= \frac{M}{L}\sin^2\left(\frac{\theta}{2}\right)\int_0^L l^2\,dl$$

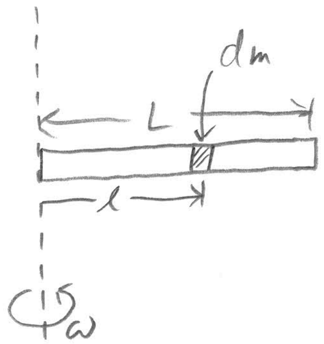
$$= \frac{M}{\cancel{L}}\sin^2\left(\frac{\theta}{2}\right)\frac{1}{3}L^{\cancel{3}\,2}$$

$$\boxed{I = \frac{1}{3}ML^2\sin^2\left(\frac{\theta}{2}\right)}$$

Rotation

A thin rod of length L has a non-uniform density profile of $\lambda = \lambda_0 \left[2\frac{l^2}{L^2} + \frac{1}{3} \right]$.

What is the moment of inertia of this rod if it's about an axis perpendicular to the light end of the rod?



$$I = \int r^2 dm$$

$$dm = \lambda dl$$

$$\Rightarrow dm = \lambda_0 \left[2\frac{l^2}{L^2} + \frac{1}{3} \right] dl$$

$r = l$ this time

$$I = \int_0^L l^2 \lambda_0 \left[2\frac{l^2}{L^2} + \frac{1}{3} \right] dl$$

$$= \lambda_0 \int_0^L \left(2\frac{l^4}{L^2} + \frac{l^2}{3} \right) dl$$

$$= \lambda_0 \left[\frac{2}{5}\frac{L^{\cancel{5}\,3}}{\cancel{L^2}} + \frac{L^3}{9} \right] = \lambda_0 L^3 \left(\frac{2}{5} + \frac{1}{9} \right) = \frac{23}{45} \lambda_0 L^3$$

$$I = \frac{23}{45} \lambda_0 L$$

$$M = \int dm = \int_0^L \lambda_0 \left[2\frac{l^2}{L^2} + \frac{1}{3} \right] dl = \lambda_0 \frac{2}{3}\frac{L^{\cancel{3}}}{\cancel{L^2}} + \frac{L}{3}$$

$$\boxed{M = \lambda_0 L}$$

So: $\boxed{I = \frac{23}{45} M L^2}$