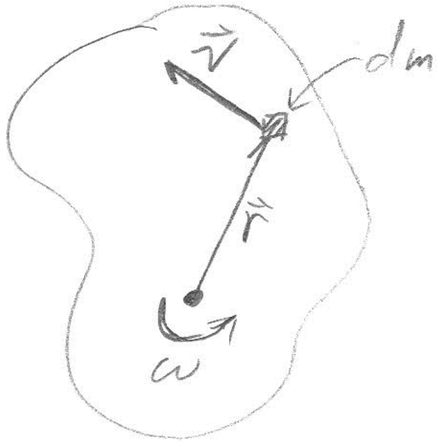
Derivation of Rotational Kinetic Energy, Rigid Body

For a point mass: $K = \frac{1}{2} m v^2$

For a rigid body:



The kinetic energy of a small piece of mass dm is:

$$dK = \tfrac{1}{2}\, dm\, v^2$$

and we integrate to get the total K:

$$\int dK = \tfrac{1}{2}\int v^2\, dm$$

But we know $v = r\omega$

So: $K = \frac{1}{2}\int r^2\omega^2\, dm$, But ω is constant (independant of r)

$$\Rightarrow K = \tfrac{1}{2}\left[\int r^2\, dm\right]\omega^2, \quad I \equiv \int r^2\, dm$$

$$\boxed{\therefore K = \tfrac{1}{2} I \omega^2 \quad \text{QED}}$$

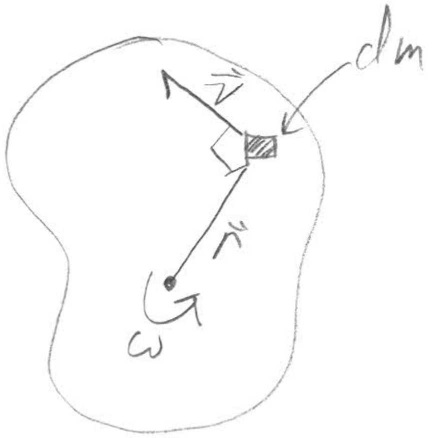
Derivation of angular momentum of a rigid body.

In general angular momentum is: $\underline{\vec{L} = \vec{r} \times m\vec{v}}$

By the definition of the cross product:

$\underline{L = mrv\sin\theta}$

For a rigid body



The angular momentum of a small piece of mass dm is:

$$dL = dm\, r v \sin\theta$$

and we integrate to get the total L.

$$\int dL = \int r v \sin\theta \, dm$$

But we know $\theta = 90°$ for all r so $\sin\theta = 1$

and $v = r\omega$ and ω is constant for all r

$$\Rightarrow L = \int r r \omega \, dm \implies L = \left[\int r^2 dm\right]\omega$$

and because $I \equiv \int r^2 dm$

$\boxed{\therefore L = I\omega}$ QED