Sample Test 5
Phys 111, Fall 2010, Section 1

1) Derivations

a) (10pts) Starting with the definition of linear Kinetic energy ($K=\frac{1}{2}mV^2$), show that rotational kinetic energy of a rigid body is $K=\frac{1}{2}I\omega^2$ where $I=\int r^2 dm$.

b) (10pts) Starting with the definition of angular momentum ($L=m|\vec{r}\times\vec{V}|$), show that the angular momentum of a rigid body is $L=I\omega$ where $I=\int r^2 dm$.

Proofs given in another post.

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2) Multiple Choice, 4 points each.

2.1) Circular disks A and B have the same mass and thickness but the density of disk A is greater than the density of disk B. Which has the greater moment of inertia? **You must explain your reasoning for full credit.**

a) Disk A
b) Disk B

IF $M_A = M_B \Rightarrow \cancel{\pi} R_A^2 \cdot \cancel{l} \rho_A = \pi R_B^2 \cdot \cancel{l} \rho_B$ l = Thickness

$$\Rightarrow \frac{R_A^2}{R_B^2} = \frac{\rho_B}{\rho_A}$$

Because $\rho_A > \rho_B$, $\frac{\rho_B}{\rho_A} < 1 \Rightarrow \frac{R_A^2}{R_B^2} < 1 \Rightarrow \boxed{R_A < R_B}$

So A has its mass close to the axis of rotation. Then $I_A < I_B$

2.2) Bar A and Bar B have the same mass, but Bar B is shorter than Bar A. Which has a larger moment of inertia? **Explain your reasoning for full credit.**

a) Bar A
b) Bar B

Bar B has more mass close to its axis of rotation, so Bar A has a greater I

2.3) Two wheels initially at rest roll the same distance *d* down an inclined plane without slipping. Wheel B has twice the radius but the same mass as wheel A. All the mass is concentrated at their rims (they are thin hoops). Which wheel is going faster at the bottom of the ramp?

a) Wheel A
~~b) Wheel B~~
c) Their speeds are the same.
d) We need more information to tell.

FBD

NSL

$mg\sin\theta - F_f = ma \Rightarrow mg\sin\theta - \frac{1}{2}ma = ma$

$\boxed{g\sin\theta = \frac{3}{2}a}$

No R!

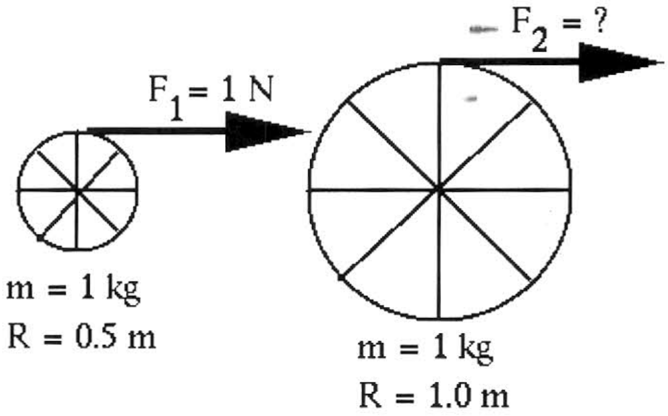
$F_f R = I\alpha$

$F_f = \frac{I}{R}\alpha$

$F_f = \frac{I}{R^2}a = \frac{\frac{1}{2}M\cancel{R^2}}{\cancel{R^2}}a$

2.4 Two wheels with fixed axles each have a mass of 1.0 kg. All of the mass is concentrated at the rim so that $I = mR^2$ for each. What does F_2 have to be for both wheels to have the same angular acceleration?

a) 0.5 N
b) 1.0 N
c) 2.0 N
d) 4.0 N



$\tau = I\alpha$

$\Rightarrow FR = mR^{\cancel{2}}\alpha$

$\Rightarrow \alpha = \dfrac{F}{mR}$

I want

$\alpha_1 = \alpha_2$

$\Rightarrow \dfrac{F_1}{\cancel{m}R_1} = \dfrac{F_2}{\cancel{m}R_2} \Rightarrow F_2 = F_1 \dfrac{R_2}{R_1} = 1 \dfrac{1}{1/2} = \boxed{2}$

2.5 A boy and a girl are riding on a merry-go-round that is turning. The boy is twice as far from the merry-go-round's center as the girl. The boy and the girl have the same mass. Which statement is true about the boy's moment of inertia with respect to the axis of rotation.

a) It is four times the girl's.
b) It is twice the girl's.
c) It is the same for both.
d) The boy has greater moment of inertia but it is impossible to say how much more than the girl's it is.

$I = mR^2$ $\quad m_B = m_G, \quad R_B = 2R_G$

$\dfrac{I_B}{I_G} = \dfrac{\cancel{m_B} R_B^2}{\cancel{m_G} R_G^2} = \dfrac{4 \cancel{R_G^2}}{\cancel{R_G^2}} = \underline{4}$

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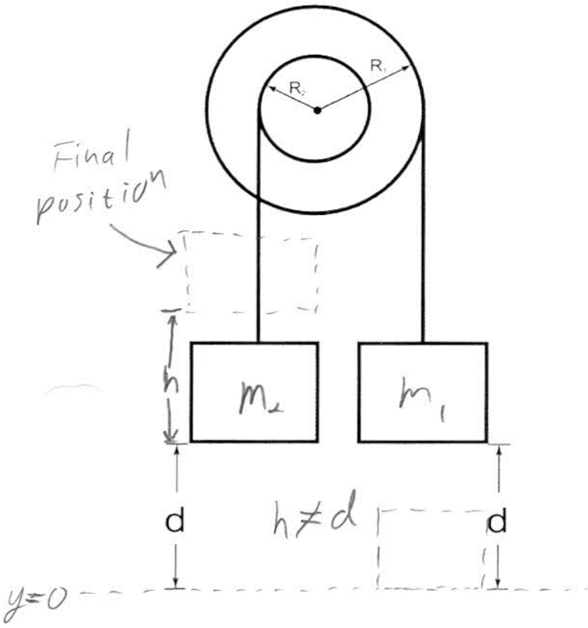
3. The picture below shows a modified atwood machine composed of two pulleys of different radii that have been glued together so that their angular velocities will be the same. Two blocks of equal mass are attached to the system by ropes. One rope is wound around the small pulley and the other rope is wound around the large pulley. The mass of the pulley is the same as the mass of the two blocks and $\boxed{R_2 = \frac{1}{2} R_1.}$

Assume that the moment of inertia of the pulley is $I = \frac{1}{2} M R_1^2$

a) If the masses are initially at rest, which way will the pulley rotate, clockwise or counter clockwise?
b) Using Work/Energy techniques, find an expression for the angular velocity of the pulleys after the mass attached to the large pulley has moved a distance d.



a) Imagine holding the pully so it can't rotate. Which will have the greater torque?

$m_1 = m_2$ and $F_T = mg$ (since nothing is moving...)

$T_1 = mg\,R_1$ and $T_2 = mg\,R_2$

$T_1 > T_2$ so it rotates clockwise.

b) $U_I = mgd + mgd$ $\qquad K_I = 0$

These are different

$U_F = mg(h+d)$ $\qquad K_F = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2 + \frac{1}{2} I \omega^2$

Relate h to d

$d = R_1\theta$, $h = R_2\theta$; θ is the same for both.

$$\frac{h}{d} = \frac{R_2\cancel{\theta}}{R_1\cancel{\theta}} \Rightarrow h = \frac{R_2}{R_1} d = \frac{\frac{1}{2}\cancel{R_1}}{\cancel{R_1}} d \Rightarrow \boxed{h = \frac{1}{2} d}$$

continued ↓

Conserve Energy

$$U_I + \cancel{K_I}^{\,0} + \cancel{W_{NCF}}^{\,0} = U_F + K_F$$

$$2mgd = mg(h+d) + \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}mv_2^2 + \tfrac{1}{2}I\omega^2$$

Plug in $h = \frac{1}{2}d$, $v_1 = R_1\omega$, and $v_2 = R_2\omega$, $I = \frac{1}{2}mR_1^2$

$$2\cancel{m}gd = \cancel{m}g\left(\tfrac{1}{2}d + d\right) + \tfrac{1}{2}\cancel{m}R_1^2\omega^2 + \tfrac{1}{2}\cancel{m}\underbrace{R_2}_{\frac{1}{2}R_1}{}^2\omega^2 + \tfrac{1}{2}\left[\tfrac{1}{2}\cancel{m}R_1^2\right]\omega^2$$

$$\Rightarrow 2gd - \tfrac{3}{2}gd = \left[\tfrac{1}{2}R_1^2 + \tfrac{1}{2}\left(\tfrac{1}{2}R_1\right)^2 + \tfrac{1}{4}R_1^2\right]\omega^2$$

$$\Rightarrow \tfrac{1}{\cancel{2}}gd = \tfrac{7}{\cancel{8}_{4}}R_1^2\omega^2$$

$$\Rightarrow \boxed{\omega^2 = \tfrac{4}{7}\frac{gd}{R_1^2}}$$

Units? $\omega^2 = \dfrac{\cancel{m}/s^2\,\cancel{m}}{\cancel{m^2}} = 1/s^2$ yay!

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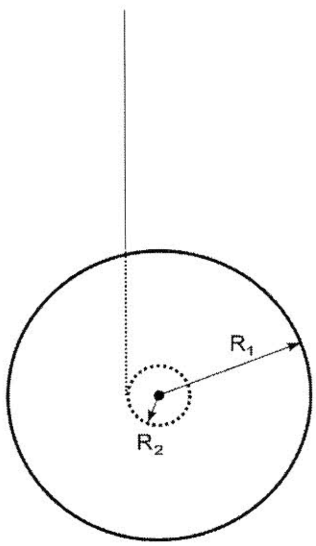
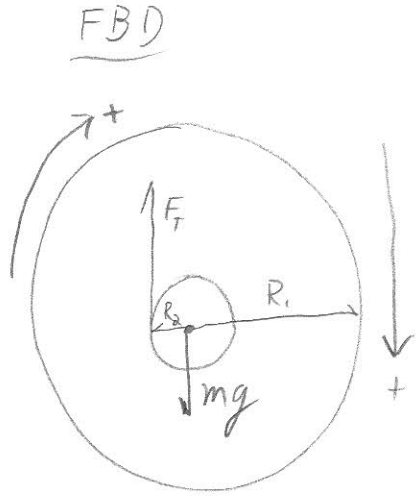
4. The picture below represents the side view of a yo-yo. The inner dashed circle represents the axle that the string is wound around. The top of the string is held stationary and the yo-yo is allowed to fall, unwinding the string as it descends.

The moment of inertia of the yo-yo is: $I_{cm} = \frac{1}{2} M R_1^2$

Use **Torque/Kinematics** to answer the following question.

If the yo-yo starts from rest, what is its angular velocity after a length of string, *d*, is unwound?





NSL

$\Sigma F = ma$

$\boxed{mg - F_T = ma}$ ①

$\Sigma \tau = I\alpha$

$\boxed{F_T R_2 - \cancel{mg(0)} = I\alpha}$ ②

We need the linear acceleration, a, of the yo-yo. Let's eliminate F_T from equations ① and ②, then solve for a:

From ①: $F_T = mg - ma$

into ②: $\boxed{(mg - ma)R_2 = I\alpha}$ ③

Now, α and a are related through $\underline{R_2}$ (not R_1).

$a = R_2\alpha$ (why? Think about it ...)

So: $(mg - ma) = \frac{I}{R_2}\frac{a}{R_2} \Rightarrow mg = \left(m + \frac{I}{R_2^2}\right)a \Rightarrow \boxed{a = \frac{m}{m + I/R_2^2}g}$

Plug in I: $a = \frac{\cancel{m}}{\cancel{m} + \frac{1}{2}\cancel{m}\frac{R_1^2}{R_2^2}} \Rightarrow a = \frac{1}{1 + \frac{1}{2}\frac{R_1^2}{R_2^2}}g$ continued

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$$\boxed{a = \frac{R_2^2}{R_1^2 + \frac{1}{2}R_2^2}\, g} \quad ④$$

Now, kinematics will provide the <u>linear</u> velocity after traveling a distance <u>d</u>.

$$y = y_0 + v_0 t + \tfrac{1}{2}at^2 \qquad v = v_0 + at$$

$$d = 0 + 0 + \tfrac{1}{2}at^2 \qquad v = 0 + at$$

$$\Rightarrow \boxed{t = \frac{v}{a}}$$

$$\Rightarrow d = \tfrac{1}{2}\cancel{a}\frac{v^2}{a^{\cancel{2}}} \Rightarrow \boxed{v = (2da)^{1/2}}$$

Now; plug in a:

$$\boxed{v = \left[2d\, \frac{R_2^2}{R_1^2 + \frac{1}{2}R_2^2}\, g\right]^{1/2}}$$

And finally, convert to angular velocity: $v = R_2\omega$

$$\Rightarrow \omega = \frac{1}{\cancel{R_2}}\left[2dg\, \frac{\cancel{R_2^2}}{R_1^2 + \frac{1}{2}R_2^2}\right]^{1/2}$$

$$\boxed{\omega = \left[\frac{2dg}{R_1^2 + \frac{1}{2}R_2^2}\right]^{1/2}}$$

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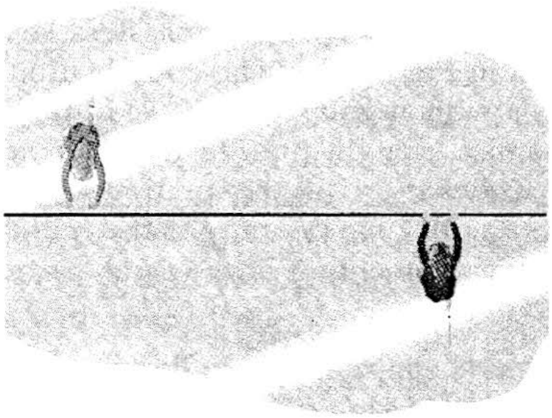
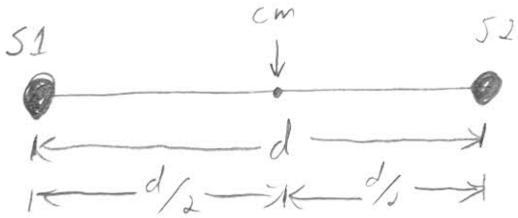
4) Two skaters, each with a mass of 50 kg, approach each other along parallel paths separated by 3.0 m. They have equal and opposite velocities of 1.4 m/s. The first skater is holding one end of a long pole with negligible mass. As the skaters pass, the second skater grabs the other end of the pole. Assume that the ice is completely frictionless.

a) What is the moment of inertia about the center of mass of the resulting skater-pole system?
b) What is the resulting angular velocity of the skater-pole system?



$m = 50\ kg$
$v = 1.4\ m/s$
$d = 3.0\ m$

a)



$$I = \sum m_i r_i^2$$

$$I = m\left(\frac{d}{2}\right)^2 + m\left(\frac{d}{2}\right)^2 = \boxed{\tfrac{1}{2} m d^2}$$

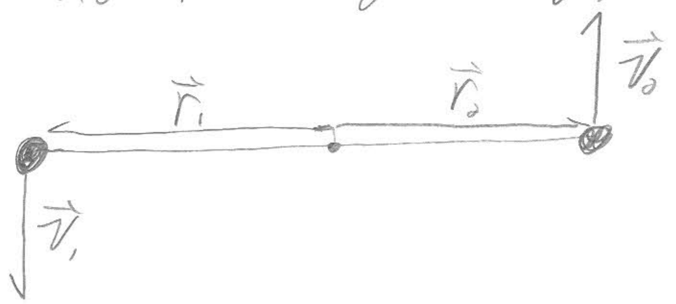
b) $L_I = L_F$

The skaters will rotate about the center of mass. But, prior to grabbing the pole, They are not a rigid body.

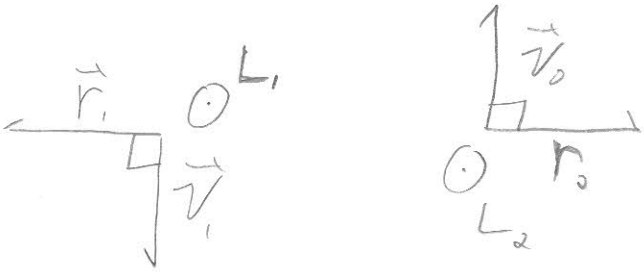
$$\vec{r}_1 \times m\vec{v}_1 + \vec{r}_2 \times m\vec{v}_2 = I\omega$$

continued ↓

At the instant of pole grabbing, it looks like this:



$\vec{r}_1 \times \vec{v}_1$ and $\vec{r}_2 \times \vec{v}_2$ give the same sign for $\vec{L}$



Both are out of the page.

So:

$$m r_1 v_1 + m r_2 v_2 = I\omega, \quad r_1 = r_2 = \frac{d}{2}$$

$$v_1 = v_2 = v$$

$$\frac{1}{2} m d v + \frac{1}{2} m d v = I\omega$$

$$m d v = I\omega \Rightarrow \omega = \frac{m d v}{I}$$

And plug in I from part a:

$$\omega = \frac{\cancel{m} d v}{\frac{1}{2}\cancel{m} d^{\cancel{2}}} \Rightarrow \boxed{\omega = 2\frac{v}{d}}$$

$$\boxed{\omega = 2\frac{1.4}{3} = 0.27 \text{ rad/s}}$$