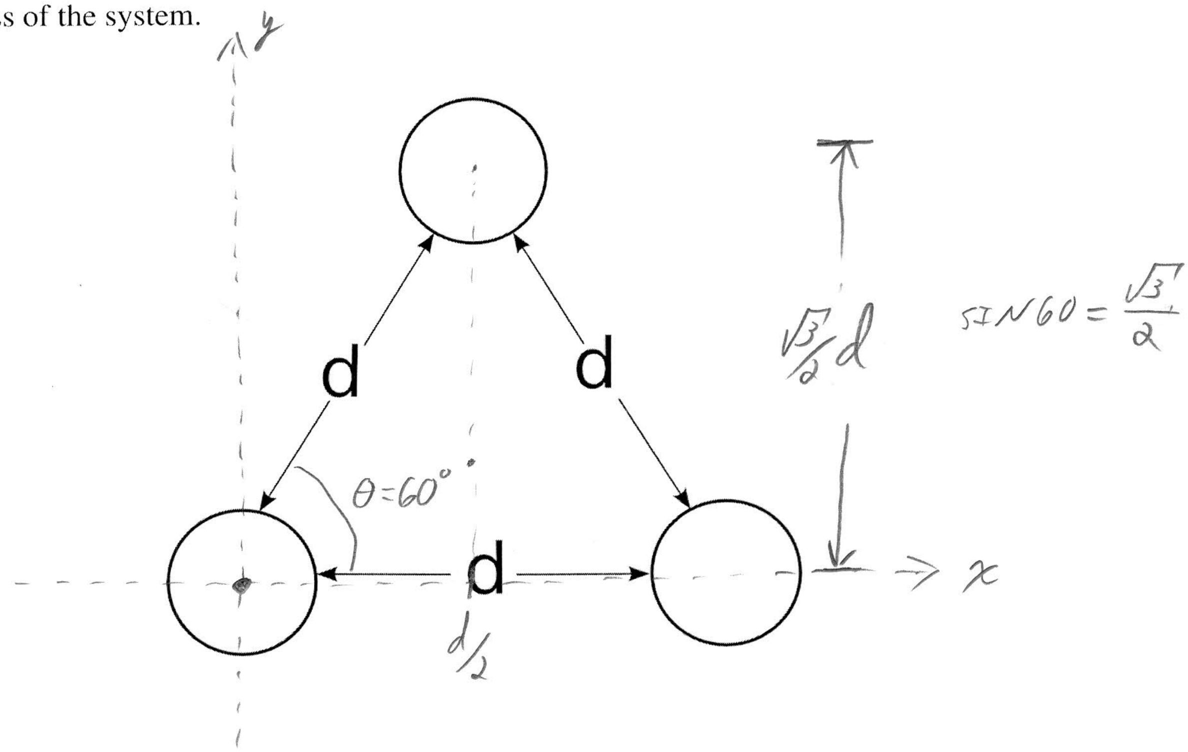
# Systems of Particles – Set 1

Three balls of equal mass form an equilateral triangle. Assume that the balls are point masses and find the center of mass of the system.




$$x: \quad x_{cm} = \frac{\sum x_i m_i}{\sum m_i} = \frac{0 \cdot m + \frac{d}{2}m + dm}{m + m + m} = \frac{\cancel{m}d(1+\frac{1}{2})}{3\cancel{m}} = \frac{3/2}{3}d$$

$$\boxed{x_{cm} = \tfrac{1}{2}d}$$

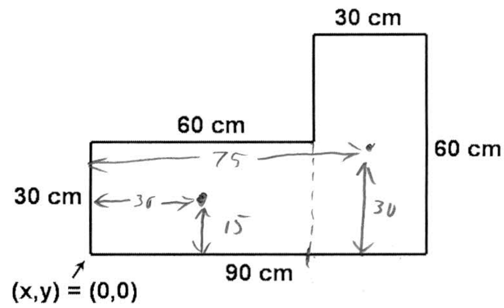
$$y: \quad y_{cm} = \frac{\sum y_i m_i}{\sum m_i} = \frac{0 \cdot m + 0 \cdot m + \frac{\sqrt{3}}{2}d}{m + m + m} = \frac{\sqrt{3}}{2}\frac{1}{3}d = \frac{d}{2\sqrt{3}}$$

$$\boxed{y_{cm} = 0.3d}$$

# Systems of Particles – Set 1

2

The uniform sheet of plywood in the figure has a mass of 20 kg. Find the x and y coordinates of the center of mass.




HINT: For a perfectly rectangular piece of plywood, the center of mass is exactly in the center of the piece. Imagine that the plywood is composed of two rectangular pieces and then treat those two pieces as point masses located at the center of mass of each piece.

Both pieces are 60 x 30, and so have mass 10 kg each

$$x_{cm} = \frac{\sum x_i m_i}{\sum m_i} = \frac{(30)(10)+(75)(10)}{10+10} = \frac{300+750}{20} = \underline{52.5\text{ cm}}$$

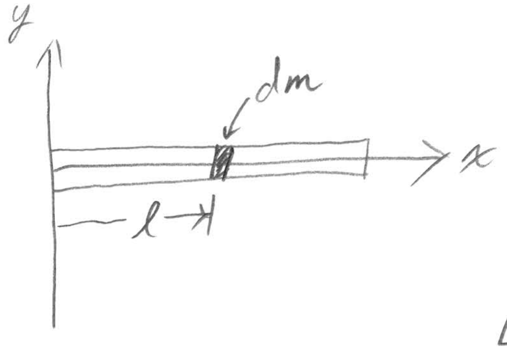
$$y_{cm} = \frac{\sum y_i m_i}{\sum m_i} = \frac{(15)(10)+(30)(10)}{10+10} = \frac{150+300}{20} = \underline{22.5\text{ cm}}$$

# Systems of Particles – Set 1

A thin rod of length L has a non-uniform density profile of $\lambda = \lambda_0 \left[ 2\frac{l^2}{L^2} + \frac{1}{3} \right]$.

where l is the distance from one end of the rod and $\lambda_0$ is a constant and has units of mass per unit length.

a) Calculate the total mass of the rod.
b) Calculate the center of mass of the rod.




$$dm = \lambda\, dl = \lambda_0 \left[ 2\frac{l^2}{L^2} + \frac{1}{3} \right] dl$$

So: $$M = \int dm = \int_0^L \lambda_0 \left[ 2\frac{l^2}{L^2} + \frac{1}{3} \right] dl = \lambda_0 \left( \frac{2}{3}\frac{L^{\cancel{3}}}{\cancel{L^2}} + \frac{1}{3}L \right)$$

$$\boxed{M = \lambda_0 L}$$

and: $$x_{cm} = \frac{1}{M}\int l\, dm = \frac{1}{M}\int_0^L l\, \lambda_0 \left[ 2\frac{l^2}{L^2} + \frac{1}{3} \right] dl$$

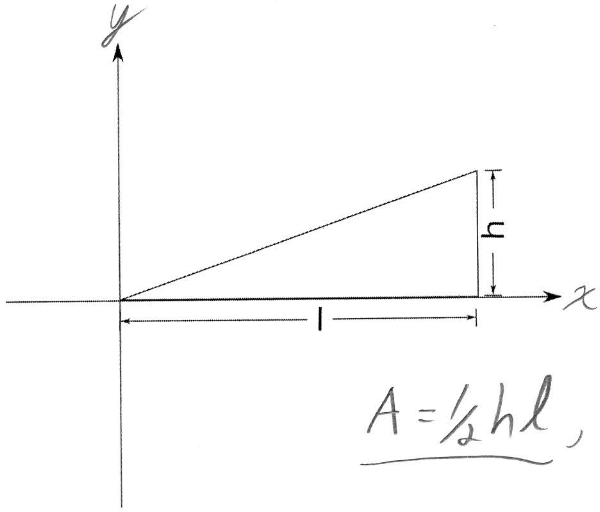
$$= \frac{1}{M}\int_0^L \lambda_0 \left[ 2\frac{l^3}{L^2} + \frac{l}{3} \right] dl = \frac{\lambda_0}{M}\left( \frac{2}{4}\frac{L^{\cancel{4}\,2}}{\cancel{L^2}} + \frac{1}{6}L^2 \right)$$

$$= \frac{\lambda_0 L^2}{M}\left( \frac{1}{\cancel{4}} + \frac{1}{6} \right) = \frac{1}{12}\frac{\lambda_0 L^2}{M}, \quad \text{But } M = \lambda_0 L$$

$$\boxed{\text{So: } x_{cm} = \frac{1}{12}L}$$

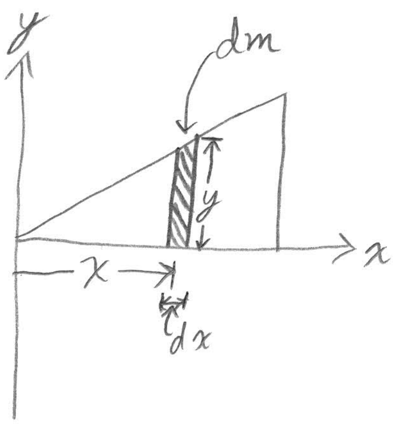
# Systems of Particles – Set 1

Calculate the center of mass of a triangular chunk of aluminum of mass *M*, length *l*, and height *h*.




$A = \frac{1}{2} h l$, $\quad \sigma = \frac{M}{A} = \frac{M}{\frac{1}{2} h l}$

## x - axis




$$x_{cm} = \frac{1}{M} \int x \, dm$$

We have to write dm in terms of x.

So we'll add up strips of mass.

What's area of the strip?

It's a rectangle with height y and width dx.

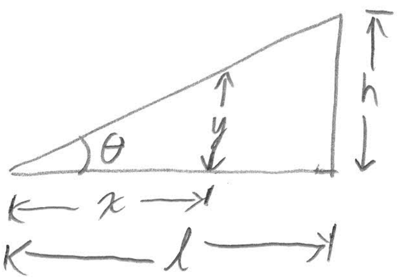
The piece has density $\sigma$ (units mass/Area)

So: $dm = \sigma y \, dx$

continued

Systems of particles 1, P4 continued.

But, $y$ varies with $x$. How?




$$\tan\theta = \frac{h}{l} = \frac{y}{x}$$

$$\Rightarrow \underline{y = \frac{h}{l}x}$$

So FINALLY,

$$dm = \sigma y\, dx$$

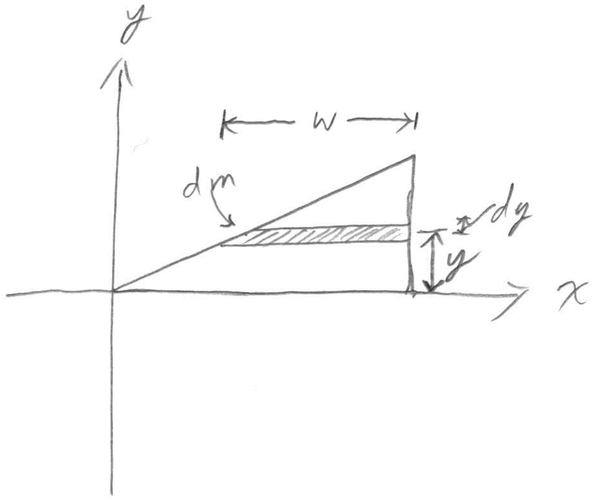
$$\Rightarrow dm = \frac{M}{\frac{1}{2}\cancel{h}l}\frac{\cancel{h}}{l}x\,dx$$

and

$$x_{cm} = \int_0^l x\, dm$$

$$\Rightarrow x_{cm} = \frac{1}{\cancel{M}}\int_0^l x\frac{2\cancel{M}}{l^2}x\,dx = \frac{2}{l^2}\int_0^l x^2\,dx = \frac{2}{3}\frac{l^{\cancel{3}}}{\cancel{l^2}}$$

$$\boxed{x_{cm} = \tfrac{2}{3}l}$$

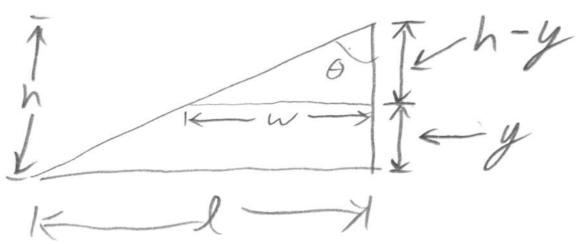



$$y_{cm} = \frac{1}{M}\int y\, dm$$

So, what's the area of the strip.

$A = w\,dy$ so $\underline{dm = \sigma w\, dy}$

what's $w$?




$$\tan\theta = \frac{l}{h} = \frac{w}{h-y}$$

$$\Rightarrow w = \frac{l}{h}(h-y)$$

$\sigma$ is the same as last time so...

$$dm = \frac{2M}{h\cancel{l}}\,\frac{\cancel{l}}{h}(h-y)\,dy$$

$$y_{cm} = \frac{1}{\cancel{M}}\int_0^h y\,\frac{2\cancel{M}}{h^2}(h-y)\,dy = \frac{2}{h^2}\int_0^h (hy - y^2)\,dy$$

$$= \frac{2}{\cancel{h^2}}\left(\frac{1}{2}h^{\cancel{3}} - \frac{1}{3}h^{\cancel{3}}\right) = h\left(1 - \tfrac{2}{3}\right) = \tfrac{1}{3}h$$

$$\boxed{y_{cm} = \tfrac{1}{3}h}$$

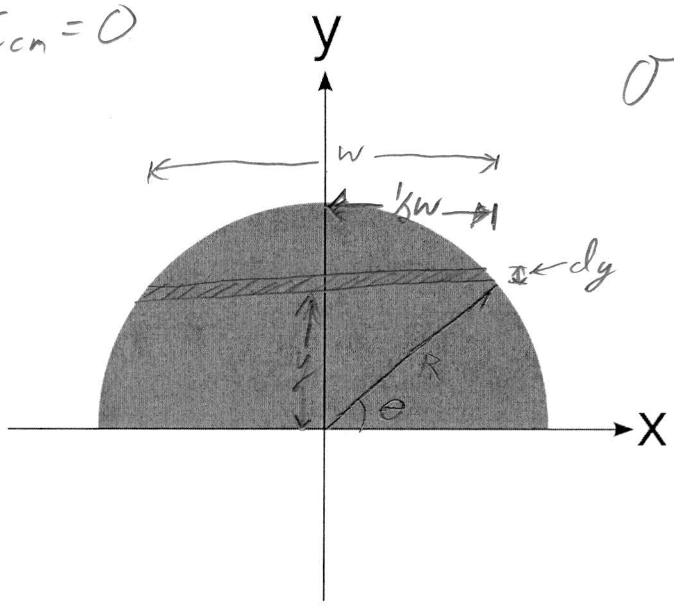
# Systems of Particles – Set 1

6 5

A flat piece of aluminum is cut into a semi-circle of radius R. Calculate the coordinates of the center of mass.

By symmetry, $x_{cm} = 0$

$$\sigma = \frac{M}{A} = \frac{M}{\frac{1}{2}\pi R^2}$$




$$y_{cm} = \frac{1}{M}\int y\,dm, \quad dm = \sigma w\,dy$$

What's $w$?

Well, according to pythagoras, $y^2 + (\frac{1}{2}w)^2 = R^2$

$$\Rightarrow \frac{1}{4}w^2 = R^2 - y^2$$

$$\Rightarrow w = 2\sqrt{R^2 - y^2}$$

Okay

$$dm = \frac{2M}{\pi R^2}\, 2\sqrt{R^2 - y^2}\, dy$$

continued ↓

Systems of Particles, Set 1, ~~P6~~ P5 continued

$$y_{cm} = \frac{1}{\cancel{M}}\int_0^R y \frac{4\cancel{M}}{\pi R^2}\sqrt{R^2-y^2}\,dy = \frac{4}{\pi R^2}\int_0^R y\sqrt{R^2-y^2}\,dy$$

hmmm... how about a substitution?

like $u = R^2 - y^2$, then $\frac{du}{dy} = -2y \Rightarrow \underline{dy = -\frac{du}{2y}}$

when: $y = R,\ u = 0$

$y = 0,\ u = R^2$

So $$y_{cm} = -\frac{2\cancel{4}}{\pi R^2}\int_{R^2}^{0} \cancel{y}(u)^{1/2}\frac{du}{2\cancel{y}} = \frac{2}{\pi R^2}\int_0^{R^2} u^{1/2}\,du$$

$$= \frac{2}{\pi R^2}\left(\frac{2}{3}u^{3/2}\right)\Bigg|_0^{R^2} = \frac{4}{3}\frac{1}{\pi R^2}(R^2)^{3/2} = \frac{4}{3}\frac{R^3}{\pi R^2}$$

$$\boxed{y_{cm} = \frac{4}{3\pi}R}$$