Energy Problems – Set ~~4~~ 5

A satellite in a perfectly circular orbit is in *Uniform Circular Motion.*

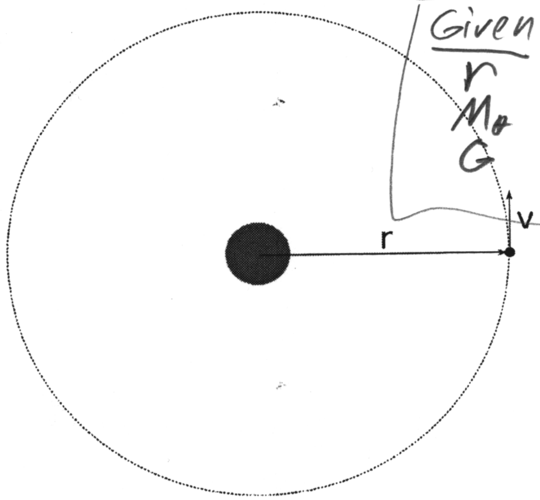
a) Using Newton's Second Law and the problem solving techniques from last unit, find an expression for the orbital velocity of a satellite a distance r from the center of the Earth.

Satellites in geosynchronous orbit always remain above the same geographic spot on the Earth's surface, making them extremely handy for communications.

b) Calculate the radius of the geosynchronous orbit.

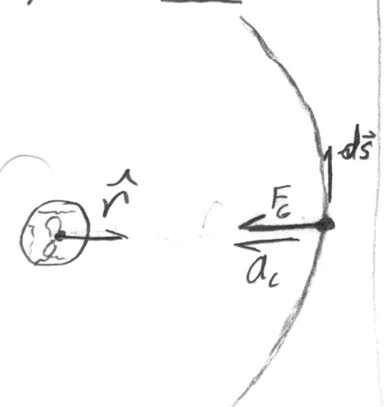
c) Calculate the orbital velocity of the geosynchronous orbit.

The radius of the Earth is: $R_{Earth} = 6.37 \times 10^{6}$ m.
The mass of the Earth is : $M_{Earth} = 5.97 \times 10^{24}$ kg
The gravitaional constant: $G = 6.67 \times 10^{-11}$ m^3/(kg s^2)



Given	Want
r	v
$M_\oplus$	
G	

a) FBD



The only force is Gravity

NSL

$$\vec{F} = m\vec{a}$$

$$-\frac{G M_\oplus m}{r^2}\hat{r} = -m a_c \hat{r}, \quad a_c = \frac{v^2}{r}$$

Uniform circular motion

Both negative

$$\Rightarrow +\frac{G M_\oplus \cancel{m}}{r^{\cancel{2}}} = +\cancel{m}\frac{v^2}{\cancel{r}}$$

$$\Rightarrow \boxed{v = \left[\frac{G M_\oplus}{r}\right]^{1/2}}$$ ✓

continued

↓

Energy Set #5, P1 continued

b) From part a we have a relationship between the orbital radius and the orbital velocity:

$$\boxed{v^2 = \frac{GM_\oplus}{r}} \quad ①$$

In this case, v and r are unknown, so we need another equation.

We are given (indirectly) the orbital Period, $T = 24\,hr$.

The total distance it travels in that time is $d = 2\pi r$

so: $d = vt \Rightarrow \boxed{2\pi r = vT}$ ② $\Rightarrow v = \frac{2\pi r}{T}$

combining ① and ②:

$$\frac{4\pi^2 r^2}{T^2} = \frac{GM_\oplus}{r} \Rightarrow \boxed{r^3 = \frac{GM_\oplus}{4\pi^2} T^2}$$

$$r = \left[\frac{(6.67\times10^{-11})(5.97\times10^{24})}{4\pi^2}(24\cdot3600)^2\right]^{1/3} = \boxed{4.22\times10^7\,m}$$

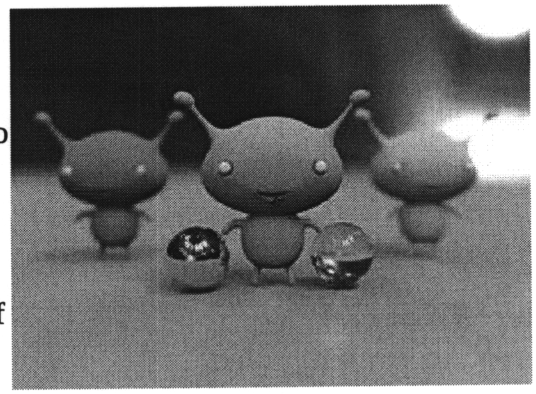
c) Solve ② for r: $r = \frac{vT}{2\pi}$, plug into ①:

$$v^2 = \frac{GM_\oplus}{vT}2\pi \Rightarrow \boxed{v^3 = \frac{2\pi GM_\oplus}{T}}$$

$$v = \left[\frac{2\pi(6.67\times10^{-11})(5.97\times10^{24})}{(24)(3600)}\right]^{1/3} = 3.07\times10^3\,m/s$$

Energy Problems – Set ~~4~~ 5

The Zeronians live on a planet with a mass of 5.0×10^{23} kg, a radius of 3.0×10^6 m. Their planet is rapidly running out of atmosphere (so there's no wind resistance) and, because they enjoy skydiving AND breathing, they have built a space-craft to leave.



a) Their space-craft, which weighs 10 kg (the zeronians are tiny), launched with an initial velocity of 3000 m/s. What will be its velocity at when it is 4.0×10^6 m from the center of the planet?

b) Prove that the maximum altitude of a space craft launched with an initial velocity V_0 from the surface of a planet is:

$$H_{max} = \frac{2G R_p M_p}{2GM_p - R_p V_0^2}$$

where R_p is the radius of the planet, M_p is the mass of the planet, and G is the gravitational constant.

c) If the ship launches with a low V_0, it will go up to H_{max}, stop and come back down. As V_0 increases, H_{max} increases. Calculate the required initial velocity for the particle to go up and NOT come back down. (HINT: What is H_{max} if the particle never comes back?)

a) Given

$M_p = 5.0 \times 10^{23}$ kg

$R_p = 3.0 \times 10^6$ m

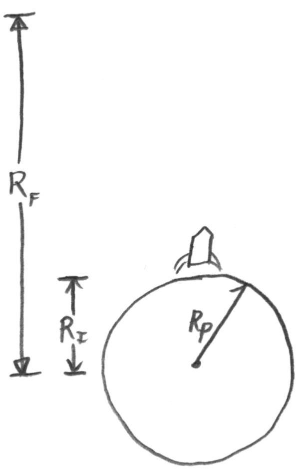
$M_s = 10$ kg

$V_0 = 3 \times 10^3$ m/s

$R_F = 4.0 \times 10^6$ m

Want

V_F



$U_I = -\frac{GM_p m_s}{R_p}$

$U_F = -\frac{GM_p m_s}{R_F}$

$K_I = \frac{1}{2} m_s V_0^2$

$K_F = \frac{1}{2} m_s V_F^2$

$$-\frac{GM_p \cancel{m_s}}{R_p} + \frac{1}{2}\cancel{m_s} V_0^2 = -\frac{GM_p \cancel{m_s}}{R_F} + \frac{1}{2}\cancel{m_s} V_F^2$$

$$\Rightarrow V_F = \left[2GM_p\left(\frac{1}{R_F} - \frac{1}{R_p}\right) + V_0^2\right]^{1/2}$$

continued

↓

Energy Problems Set 5, P3 - continued

$$\Rightarrow v_F = \left[\left(6.67\times10^{-11}\frac{m^3}{kg\cdot s^2}\right)\left(5.0\times10^{23}kg\right)\left(\frac{1}{4\times10^6 m} - \frac{1}{3\times10^6 m}\right) + \left(3\times10^3 m/s\right)^2\right]^{1/2}$$

$$\Rightarrow \boxed{v_F = 2.49\times10^3\ m/s}$$

b) Maximum altitude when $v_F = 0$

$$U_I = -\frac{GM_p m_s}{R_p} \qquad K_I = \tfrac{1}{2}m_s v_0^2$$

$$U_F = -\frac{GM_p m_s}{H_{max}} \qquad K_F = 0$$

$$\Rightarrow -\frac{GM_p \cancel{m_s}}{R_p} + \tfrac{1}{2}\cancel{m_s}v_0^2 = -\frac{GM_p\cancel{m_s}}{H_{max}}$$

$$\Rightarrow \frac{1}{H_{max}} = \frac{1}{R_p} - \frac{v_0^2}{2GM_p}$$ ← Put this over a common denominator

$$\Rightarrow \frac{1}{H_{max}} = \frac{2GM_p - R_p v_0^2}{2GM_p R_p}$$

$$\Rightarrow \boxed{H_{max} = \frac{2GM_p R_p}{2GM_p - R_p v_0^2}}$$

c) "Escape" when $v_F \geq 0$ when $H_{max} \to \infty$

Just barely escape at $v_F = 0$.

continued
↓

Energy Problems Set 5, P3 - continued.

$$U_I = -\frac{GM_p m_s}{R_p} \qquad K_I = \tfrac{1}{2} m_s v_{esc}$$

$$U_F = -\frac{GM_p m_s}{\infty} = 0 \qquad K_F = 0$$

$$\Rightarrow -\frac{GM_p \cancel{m_s}}{R_p} + \tfrac{1}{2}\cancel{m_s} v_{esc}^2 = 0$$

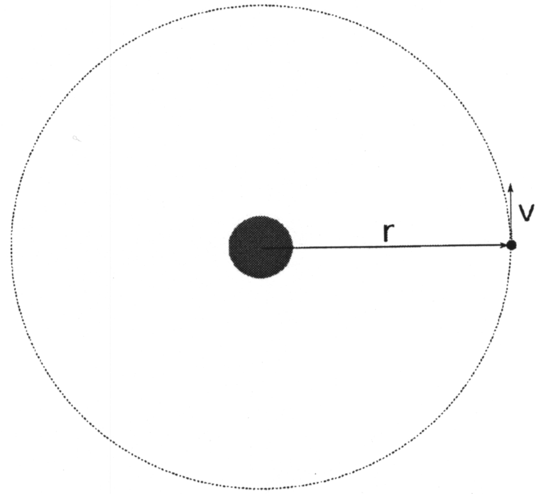
$$\Rightarrow \boxed{v_{esc} = \left[\frac{2GM_p}{R_p}\right]^{1/2}}$$

Consider a 5.0×10^4 kg satellite in a circular orbit of radius r about the Earth.

a) Find an expression for the Total Mechanical Energy of the Satellite (Potential Energy plus Kinetic Energy)

b) If the satellite is in low Earth orbit, 200 km above the Earth's surface, how much work must the satellite thrusters do to boost it to the Geosynchronous orbit?

The radius of the Earth is: $R_{Earth} = 6.37 \times 10^6$ m.
The mass of the Earth is : $M_{Earth} = 5.97 \times 10^{24}$ kg
The gravitaional constant: $G = 6.67 \times 10^{-11}\ m^3/(kg\ s^2)$



a) Given	Want
$m_s = 5 \times 10^4$ kg	$U + K = E_T$
$M_\oplus = 5.97 \times 10^{24}$ kg
r

$U = -\frac{G M_\oplus m_s}{r}$, $K = \frac{1}{2} m_s v^2$,

But what's v?

v isn't given...

$$① \quad \boxed{E_T = \frac{1}{2} m_s v^2 - \frac{G M_\oplus m_s}{r}}$$

From NSL: $F = ma \Rightarrow -\frac{G M_\oplus \cancel{m_s}}{r^{\cancel{2}}} = -\cancel{m_s} \frac{v^2}{\cancel{r}}$

$$\Rightarrow \boxed{v^2 = \frac{G M_\oplus}{r}} \quad ②$$

Combining ① and ②:

$$E_T = \frac{1}{2} m_s \frac{G M_\oplus}{r} - \frac{G M_\oplus m_s}{r} \Rightarrow \boxed{E_T = -\frac{1}{2} \frac{G M_\oplus m_s}{r}}$$

continued ↓

b) From the conservation of energy:

$$\underbrace{U_I + K_I}_{E_{TI}} + \underbrace{W_{NCF}}_{\text{Rocket}} = \underbrace{U_F + K_F}_{E_{TF}}$$

$$\Rightarrow E_{TI} + W_{Rocket} = E_{TF}$$

$$\Rightarrow W_{Rocket} = E_{TF} - E_{TI}$$

From part a

$$= -\frac{1}{2}\frac{GM_\oplus m_s}{r_F} + \frac{1}{2}\frac{GM_\oplus m_s}{r_I}$$

$$\boxed{W_{Rocket} = \frac{1}{2}GM_\oplus m_s\left[\frac{1}{r_I} - \frac{1}{r_F}\right]}$$

$$W_{Rocket} = \frac{1}{2}(6.67\times10^{-11})(5.97\times10^{24})(5.0\times10^{4})\left[\frac{1}{(6.3702\times10^{6})} - \frac{1}{(4.22\times10^{7})}\right]$$

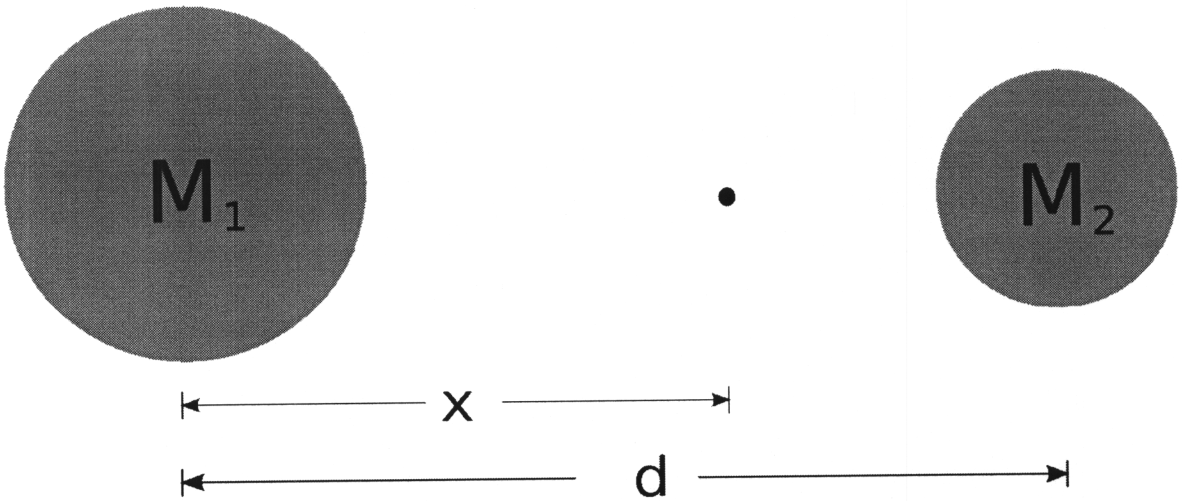
$$= \boxed{1.33\times10^{12}\text{ Joules}}$$

Imagine a spacecraft traveling from the Earth (mass M_1) to the Moon (mass M_2). The spacecraft will feel the force of gravity from both objects. Close to the Earth, the Earth's gravitational force will be greater. Close to the Moon, the Moon's gravitational force will be greater. Somewhere in between is an equilibrium point where the force goes to zero.

a) Use Newton's Universal Law of Gravitation to find the distance x where the net force goes to zero.

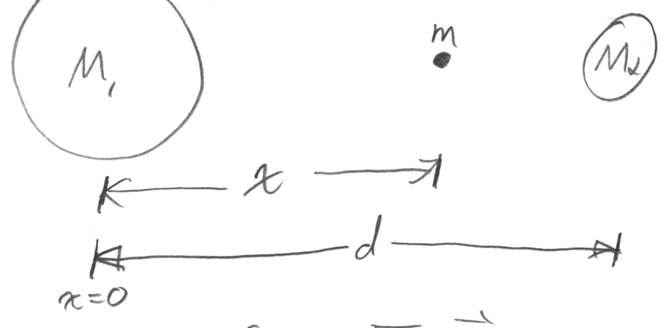
b) The gravitational potential is the sum of the potentials from each body. Writing an expression for the potential at a distance x from M_1. Find the equilibrium point by locating any minima in the function.

c) Sketch the potential function. Is the equilibrium point stable or unstable?



Solution on next page

a)



Looking for $\sum \vec{F} = 0 \Rightarrow \vec{F}_1 + \vec{F}_2 = 0$

$$\Rightarrow -\frac{GM_1 m}{x^2} + \frac{GM_2 m}{(d-x)^2} = 0$$

$$\Rightarrow \frac{M_1}{x^2} = \frac{M_2}{(d-x)^2} \Rightarrow \frac{\sqrt{M_1}}{x} = \frac{\sqrt{M_2}}{(d-x)}$$

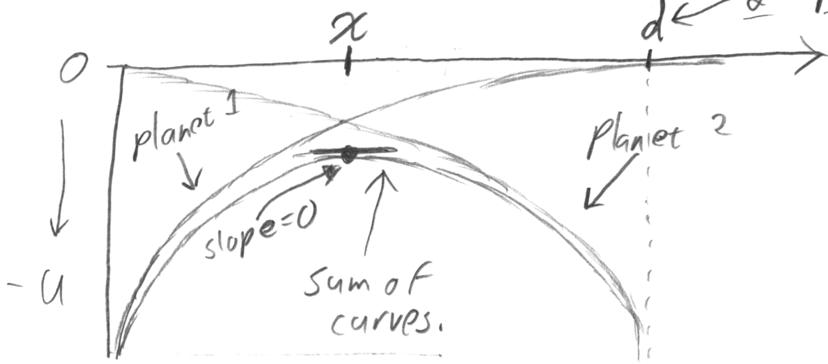
$$\Rightarrow \sqrt{M_1}\,d - \sqrt{M_1}\,x = \sqrt{M_2}\,x$$

$$\Rightarrow x\left(\sqrt{M_1} + \sqrt{M_2}\right) = \sqrt{M_1}\,d$$

$$\Rightarrow \boxed{x = \frac{\sqrt{M_1}}{\sqrt{M_1} + \sqrt{M_2}}\,d}$$

b) The total potential is due to both masses. We can simply add them. $U_G = -\frac{GMm}{r}$ goes as $-\frac{1}{r}$, always negative.

$d \leftarrow 2^{nd}$ planet here



equilibrium when slope = 0

c) Now mathematically:

$$U_T = U_{M_1} + U_{M_2} \Rightarrow U_T = -\frac{GM_1m}{x} - \frac{GM_2m}{(d-x)}$$

Let's find the extrema, when $\frac{dU_T}{dx} = 0$,

That's the equilibrium point.

$$\frac{dU_T}{dx} = \frac{d}{dx}\left[-\frac{GM_1m}{x} - \frac{GM_2m}{(d-x)}\right] = 0$$

$$= -GM_1m\frac{d}{dx}\left(\frac{1}{x}\right) - GM_2m\frac{d}{dx}\frac{1}{(d-x)} = 0$$

$$= \boxed{+\frac{GM_1m}{x^2} - \frac{GM_2m}{(d-x)^2} = 0}$$

Same as part a!!

$$\boxed{x = \frac{\sqrt{M_1}}{\sqrt{M_2}+\sqrt{M_1}}d}$$

While on its way to Jupiter in 1993, the Galileo spacecraft made a flyby of asteroid Ida. Images caputred of Ida showed that the asteriod has a tiny moon of its own, since given the name Dactyl. Measurements found Ida to be about 56x24x21 km in size and Dactyles orbital period and radius are approximately 27h and 95 km respectively. From these data and assuming that both Ida and its moon are spherical, determine Ida's approximate mass and density.

Given

$V_{IDA} = (56 \cdot 24 \cdot 21)\ km^3$

$T_{orb} = 27h$

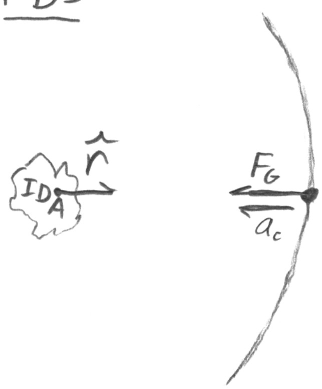
$R_{orb} = 95km$

Want

M_I

$\rho_I \quad (\rho = \text{density})$

FBD



NSL

$$\vec{F} = m\vec{a}$$

$$+\frac{G M_I \cancel{M_d}}{R_{orb}^{\cancel{2}}} = +\cancel{M_d}\frac{V_{orb}^2}{\cancel{R_{orb}}}$$

$$\Rightarrow M_I = \frac{V_{orb}^2 R_{orb}}{G}, \quad \text{and} \quad 2\pi R_{orb} = V_{orb} T_{orb}$$

$$\Rightarrow \boxed{M_I = \frac{4\pi^2 R_{orb}^3}{T_{orb}^2 G}} \Rightarrow M_I = \frac{4\pi^2 (95 \times 10^3 m)^3}{(27h \cdot 3600 s/hr)^2 (6.67 \times 10^{-11})}$$

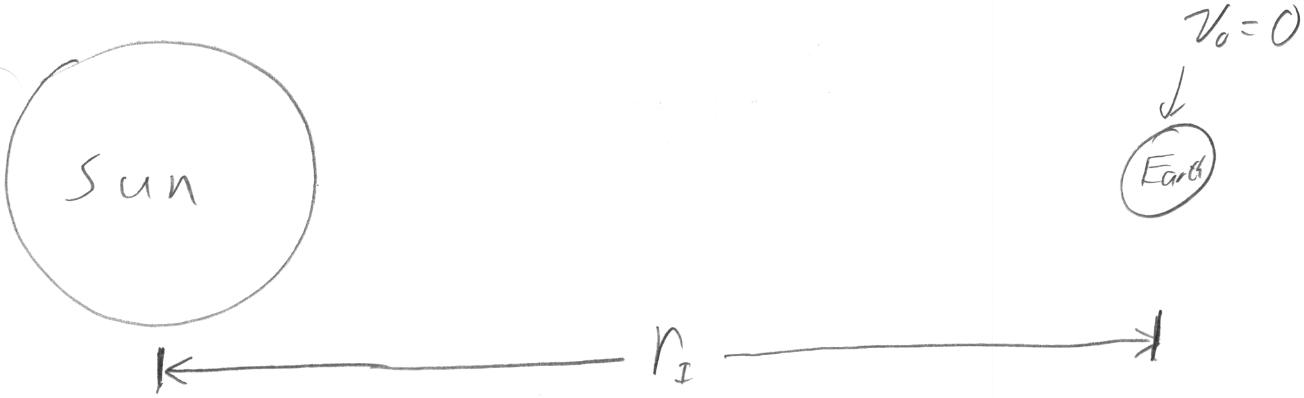
$$M_I = \boxed{5.37 \times 10^{16} kg}$$

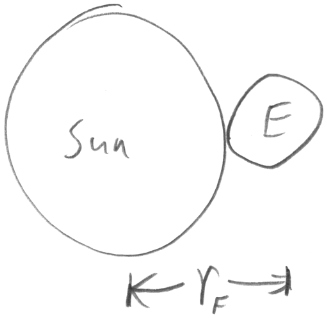
$$\rho_I = \frac{M_I}{V_I} = \frac{5.37 \times 10^{16}}{(56)(24)(21)(1 \times 10^3)^3} = \boxed{1.9 \times 10^9\ kg/km^3}$$

Suppose the Earth were suddenly to stop revolving around the Sun. The gravitational force would then pull it directly into the Sun. What would be the Earth's speed as it crashed (i.e., just when the Earth's *surface* hits the Sun's *surface*)? Assume the Earth starts at rest and the Sun doesn't move toward the earth.

$M_{sun} = 1.99\text{x}10^{30}$ kg
$R_{sun} = 695{,}500$ km
$R_{Earth} = 6378.1$ km
$M_{Earth} = 5.9742 \times 10^{24}$ kg

Solution on next page
↓





Given

$M_\odot = 1.99 \times 10^{30}$ kg

$R_\odot = 6.955 \times 10^5$ km

$R_\oplus = 6.3781 \times 10^3$ km

$M_\oplus = 5.9742 \times 10^{24}$ kg

$v_i = 0$

Want

v_F

$$U_I = -\frac{G M_\odot m_\oplus}{r_I} \qquad K_I = 0 \qquad r_I = 1.49 \times 10^8 \text{ km}$$

$$r_F = R_\odot + R_\oplus$$

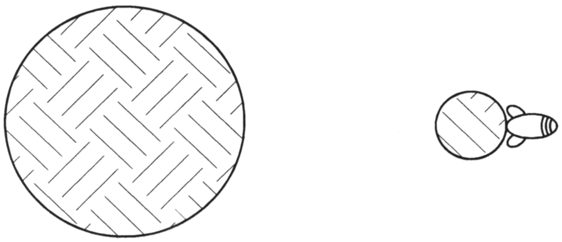
$$U_F = -\frac{G M_\odot m_\oplus}{r_F} \qquad K_F = \frac{1}{2} m_\oplus v_F^2$$

$$-\frac{G M_\odot \cancel{M_\oplus}}{r_I} = -\frac{G M_\odot \cancel{m_\oplus}}{r_F} + \frac{1}{2}\cancel{m_\oplus} v_F^2$$

$$v_F^2 = 2GM_\odot\left[\frac{1}{r_F} - \frac{1}{r_I}\right] = \left[2(6.67\times10^{-11})(1.99\times10^{30})\left[\frac{1}{7.02\times10^5} - \frac{1}{1.49\times10^8}\right]\right]^{1/2}$$

$$\boxed{v_F = 1.94 \times 10^7 \text{ m/s}}$$

Determine the escape speed of a rocket from the far side of Ganymede, the largest of Jupiter's moons. The radius of Ganymede is 2.640×10^{6} m, and its mass is 1.495×10^{23} kg. The mass of Jupiter is 1.900×10^{27} kg, and the distance between the center of mass of Ganymede and the center of mass of Jupiter is 1.071×10^{9}m. Be sure to include the gravitational effect of Jupiter, but you can ignore the motion of Jupiter and Ganymede as they revolve about the systems center of mass.



solution on next page

Energy Problems Set 5, P5 -

①

Given

$R_G = 2.64 \times 10^6$ m

$M_G = 1.495 \times 10^{23}$ kg

$M_J = 1.900 \times 10^{27}$ kg

$d = 1.071 \times 10^9$ m

Want

v_{esc}

Escape when $U_F = 0$ ($r_F = \infty$) and $K_F = 0$

$$U_I = -\underbrace{\frac{G M_J m_R}{d + R_G}}_{\text{Jupiter}} - \underbrace{\frac{G M_G m_R}{R_G}}_{\text{Ganymede}}$$

$$K_I = \frac{1}{2} m_R v_{esc}^2$$

$$K_F = 0$$

$$U_F = 0$$

$$-\frac{G M_J \cancel{m_R}}{d + R_G} - \frac{G M_G \cancel{m_R}}{R_G} + \frac{1}{2} \cancel{m_R} v_{esc}^2 = 0$$

$$\boxed{v_{esc} = \left[2G\left(\frac{M_J}{d + R_G} + \frac{M_G}{R_G}\right)\right]^{1/2}}$$

(a) Find the gravitational force of one star on the other.

(b) Find the orbital speed of each star and the period of the orbit.

(c) How much energy must be added to the system to separate the two stars to infinity, where they are at rest? Write your answer in terms of M and R. (Note: the addition of energy, however it physically happens, is tantamount to non-conservative work.)

Solution on next page

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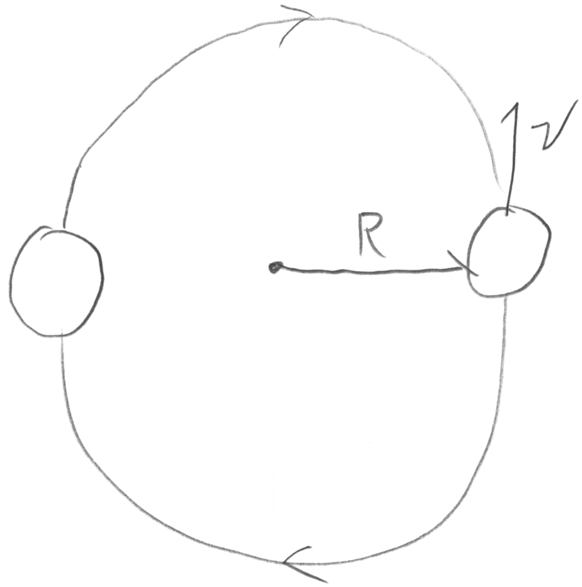
Energy Problems Set 5, P6 ①

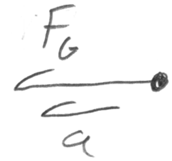
a) Given Want

M, R $\qquad$ F

$$\vec{F}_G = -\frac{G M_1 M_2}{r^2}\hat{r} \Rightarrow |\vec{F}_G| = \frac{G M^2}{4R^2}$$

b)





$$F_G = M\frac{v^2}{R}$$

$$\Rightarrow \frac{G M^{\cancel{2}}}{4R^{\cancel{2}}} = \cancel{M}\frac{v^2}{\cancel{R}}$$

$$\Rightarrow \boxed{v = \left[\frac{GM}{4R}\right]^{1/2}}$$

c) $U_I = -\frac{GM^2}{2R}$ $\qquad$ $K_I = \frac{1}{2} M v_{orb}^2 + \frac{1}{2} M v_{orb}^2$

2 stars

$U_F = 0$ $\qquad$ $K_F = 0$; $r \to \infty$

at $r = \infty$

continued ↓

Energy Problems Set 5, P6 – continued ②

$$U_I + K_I + W_{NCF} = U_F + K_F$$

$2R$

Energy to separate

$$W = \frac{GM^2}{2R} - M\boxed{\frac{GM}{4R}} \leftarrow v_{orb} \text{ from part b}$$

$$W = \frac{GM^2}{R}\left[\frac{1}{2} - \frac{1}{4}\right]$$

$$W = \frac{1}{4}\frac{GM^2}{R}$$