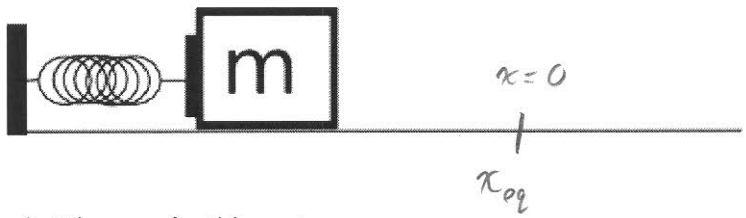
Oscillation – Set 2

A block of mass *m* is attached to a spring with spring constant *k* and allowed to move on a frictionless surface.



1) Draw a Free Body Diagram for this system.

2) Let the equilibrium position of the spring be x=0, write **Newton's Second Law**, and solve it for the acceleration of the system.

$$F = ma$$

$$-kx = ma \Rightarrow a = -\frac{k}{m}x \Rightarrow \boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

3) Show that the **Simple Harmonic Oscillator** equation satisfies the differential equation above using substitution.

SHO general solution: $x(t) = A\cos(\omega t + \phi)$

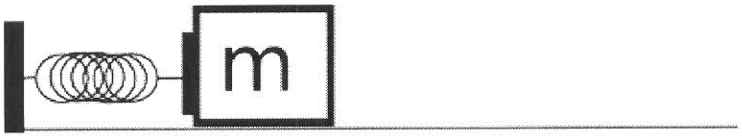
So: $\frac{d^2}{dt^2}\left[A\cos(\omega t + \phi)\right] = -\frac{k}{m} A\cos(\omega t + \phi)$

$\Rightarrow +\omega^2 \cancel{A} \cancel{\cos(\omega t + \phi)} = +\frac{k}{m} \cancel{A} \cancel{\cos(\omega t + \phi)} \Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}}$

4) What did you find for the angular frequency of the oscillator?

$$\omega = \sqrt{\frac{k}{m}}$$

A block of mass m is attached to a spring with spring constant k and allowed to move on a frictionless surface.



At $t=0$, the block is at $x=0$ (the spring's equilibrium point), and is moving to the right with a velocity $v=v_0$.

Find the amplitude A and the phase angle φ.

Initial Conditions

$x(0) = 0$

$v(0) = V_0$

so: $x(t) = A\cos(\omega t + \phi)$

$\Rightarrow 0 = A\cos(\phi)$

$\Rightarrow \phi = \cos^{-1}(0) \Rightarrow \boxed{\phi = \pi/2 \text{ or } \boxed{3\pi/2}}$

For + initial v

and $v(t) = -\omega A \sin(\omega t + \phi)$

$\Rightarrow V_0 = -\omega A \sin(3\pi/2) \Rightarrow \boxed{A = \dfrac{V_0}{\omega}}$

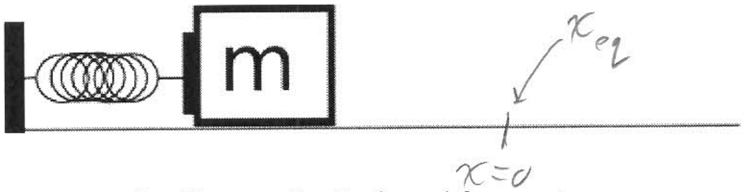
What happens to A as v_0 is increased?

A increases

What happens to φ as v_0 is increased?

no change

A block of mass m is attached to a spring with spring constant k and allowed to move on a frictionless surface.



At $t=0$, the block is compressed a distance d and released from rest.

Find the amplitude A and the phase angle φ.

Given
$x(0)=d$
$v(0)=0$

$$x(t)=A\cos(\omega t+\phi) \Rightarrow \boxed{d=A\cos(\phi)} \quad (1)$$
$$v(t)=-\omega A\sin(\omega t+\phi) \Rightarrow \boxed{0=-\omega A\sin(\phi)} \quad (2)$$
$$\text{From (2): } \sin(\phi)=0 \Rightarrow \boxed{\phi=0 \text{ or } \boxed{\pi}}$$ neg. init. displacement
$$\text{From (1): } d=A\cos(\pi) \Rightarrow \boxed{A=-d}$$

What happens to A as d is increased?

A increases

What happens to φ as d is increased?

No change

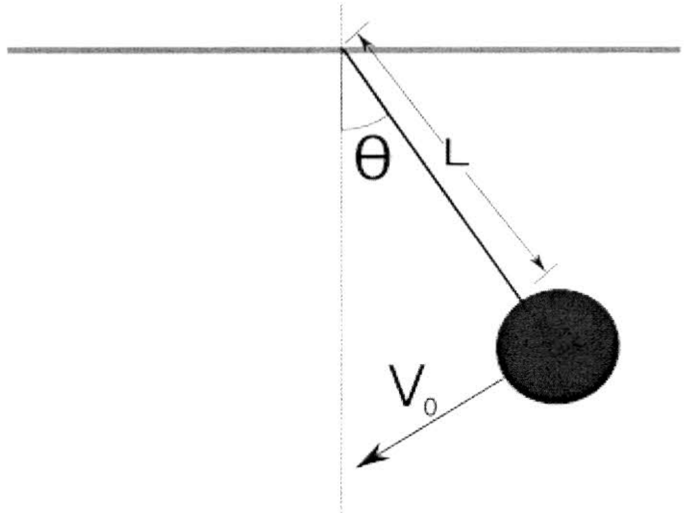
Why is φ different than it was in the last problem?

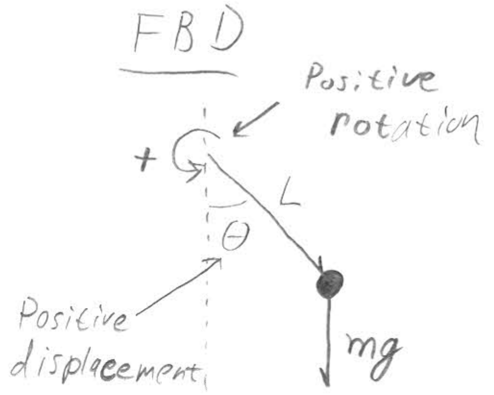
$t=0$ is at different points in the oscillator cycle.

Oscillation – Set 2

Below is a simple pendulum consisting of a massless rod of length L with a point mass if mass m attached to the end.

a) Find the frequency of small oscillations of the pendulum.
b) At t=0, the pendulum makes an angle θ_0 with the vertical and the point mass has a velocity V_0. What is ~~the amplitude of the oscillator?~~ Phase angle?





$\vec{L} \times m\vec{g} \Rightarrow \otimes$ gives negative torque by rhr

NSL

$\sum T = I\alpha, \quad I = mL^2$

$-\cancel{m}g\cancel{L}\sin\theta = \cancel{m}L^{\cancel{2}}\dfrac{d^2\theta}{dt^2}$

$\Rightarrow \dfrac{d^2\theta}{dt^2} = -\dfrac{g}{L}\sin\theta$, Almost a SHO.

For small θ, $\sin\theta \cong \theta$ (small angle approximation)

So; For small oscillations: $\boxed{\dfrac{d^2\theta}{dt^2} = -\dfrac{g}{L}\theta}$ SHO!

$\boxed{\omega = \sqrt{\dfrac{g}{L}}}$

continued ↓

Oscillation Set 2, P2 continued

b) $\theta(0) = \theta_0$, $v(0) = V_0 = \omega_0 L$

!!Be <u>very</u> careful!!

This angular velocity is <u>not</u> the same ω as the oscillator frequency, $\sqrt{g/L}$. Let's call $\boxed{\sqrt{g/L} = \omega_F}$

Angular versions of SHO general solution

$$\theta(t) = A\cos(\omega_F t + \phi)$$
$$\omega(t) = -\omega_F A \sin(\omega_F t + \phi)$$

$$\theta_0 = A\cos(\phi)$$
$$\frac{V_0}{L} = -\omega_F A \sin(\phi)$$

$$\Rightarrow \frac{V_0}{L\theta_0} = \frac{-\omega_F \cancel{A} \sin(\phi)}{\cancel{A}\cos(\phi)}$$

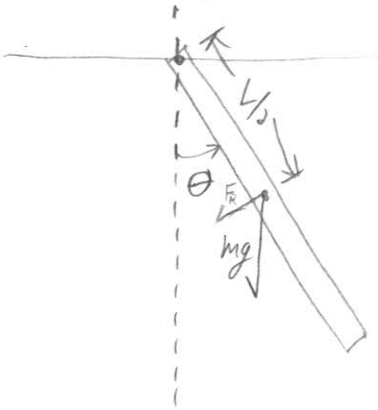
$$\Rightarrow \boxed{\tan(\phi) = -\frac{V_0}{\omega_F L \theta_0}}$$

$$\Rightarrow \tan(\phi) = -\left(\frac{L}{g}\right)^{1/2} \frac{V_0}{L\theta_0}$$ plug in ω_F

$$\Rightarrow \boxed{\tan(\phi) = -\frac{V_0}{\theta_0\sqrt{gL}}}$$

A meter stick with a mass M is suspended from one end and allowed to swing like a pendulum.

a) What is its **period** of small oscillations?
b) What length *L* does a simple pendulum (a point mass attached to a massless rod) need in order to have the same period?



$$I = \tfrac{1}{3}ML^2, \quad F = -mg\sin\theta, \quad L = 1 \text{ meter}$$

$$\sum \tau = I\alpha$$

$$-(mg\sin\theta)\frac{L}{2} = I\alpha$$

$$-mgL\sin\theta = 2I\alpha$$

$$-\cancel{m}g\cancel{L}\sin\theta = 2\cdot\tfrac{1}{3}\cancel{m}L^{\cancel{2}}\frac{d^2\theta}{dt^2}$$

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{3}{2}\frac{g}{L}\sin\theta}$$

a) For small oscillations, $\sin\theta \cong \theta$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{3}{2}\frac{g}{L}\theta \Rightarrow \omega = \left(\frac{3}{2}\frac{g}{L}\right)^{1/2}$$

and $T = \frac{2\pi}{\omega}$, $L = 1$ m, $T = 2\pi\left(\frac{2}{3g}\right)^{1/2} = \boxed{1.6\text{ s}}$

b) From problem 2, the frequency of a simple pendulum is

$$\omega = \sqrt{g/L_s} \text{ so } T = \frac{2\pi}{\omega} \Rightarrow T = 2\pi\sqrt{L_s/g}$$

We want $T_m = T_s$ so: $\cancel{2\pi}\sqrt{L_s/g} = \cancel{2\pi}\left(\frac{2L}{3g}\right) \Rightarrow \frac{L_s}{\cancel{g}} = \frac{2}{3}\frac{1}{\cancel{g}}$

meter stick — T_m; simple — T_s

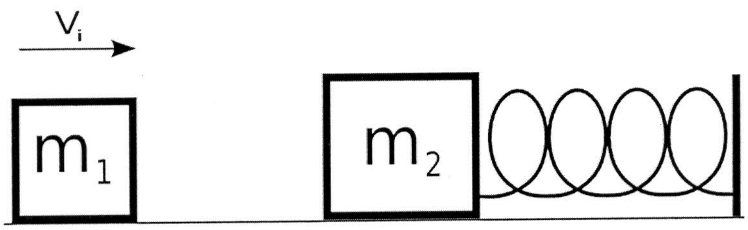
$$\boxed{L_s = \tfrac{2}{3}L}$$

(1)

Oscillation – Set 1

5

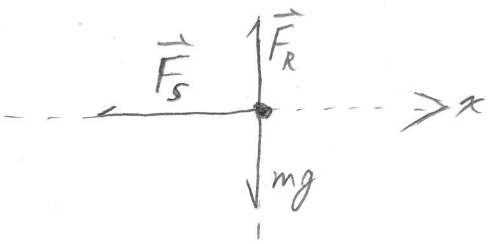
A block with a mass of m_1 = 10 kg is moving to the right with a velocity V_i. It collides and sticks to a block with a mass of m_2 = 15 kg. The second mass is attached to a spring with spring constant k=3 N/m. Before the collision, the spring is at rest in it's equilibrium position.



a) What is the frequency, ~~v~~ ω, of the resulting oscillator after the collision?
b) Assuming that the moment of collision is t=0, find the phase constant of the oscillator?
c) If the resulting amplitude of the oscillator is A = 3 m, what was the initial velocity of m_1?

Step (1) – Use NSL to find ω, the oscillator frequency
The oscillator starts after the collision so:

FBD



NSL

$$\sum \vec{F} = m\frac{d^2x}{dt^2}$$

$$-kx = m_T\frac{d^2x}{dt^2} \Rightarrow \boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x} \quad (1)$$

General solution is: $\boxed{x(t) = A\cos(\omega t + \phi)}$ (2)

so, combine (1) and (2)

$$\frac{d^2}{dt^2}\left[A\cos(\omega t + \phi)\right] = -\frac{k}{m_T}\left(A\cos(\omega t + \phi)\right)$$

$$\Rightarrow -\omega^2 \cancel{A\cos(\omega t + \phi)} = -\frac{k}{m_T}\left(\cancel{A\cos(\omega t + \phi)}\right)$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}} \Rightarrow \omega = \left(\frac{3}{(10+15)}\right)^{1/2} \Rightarrow \boxed{\omega = 0.35 \text{ rad/s}}$$

Oscillation Problems Set 1, P5 continued

Step ② – Use Physics to find the initial conditions

Let: $t=0$ be the moment of collision.

$\boxed{x(t=0)=0}$, starting position is at equilibrium.

Let's find $v(t=0)=v_F$. It's an inelastic collision

$$P_I = P_F$$

$$m_1 v_{1I} = (m_1+m_2)\, v_F$$

$$\Rightarrow \boxed{v_F = \frac{m_1}{m_1+m_2} v_{1I}}$$

Step ③ – Use the general solution of a SHO to solve the problem.

Part b asks for the phase constant.

Let's find it.

In general:

$$x(t) = A\cos(\omega t + \phi)$$

$$v(t) = -\omega A \sin(\omega t + \phi)$$

$$\Rightarrow 0 = A\cos(\phi) \Rightarrow \cos(\phi) = 0$$

True when $\phi = \pi/2,\ 3/2\,\pi$

continued

↓

Oscillation Problems Set 1, P5 continued

But we need $v(0)$ to be positive:

$$v(0) = -\omega A \sin(\phi), \quad \sin\left(\frac{\pi}{2}\right) = 1, \quad \sin\left(\frac{3\pi}{2}\right) = -1$$

so, $\boxed{\phi = 3\pi/2}$

Now, we'll use the velocity equation to find v_{1I}

$$v(0) = v_F = -\omega A \sin(\phi), \quad \phi = 3\pi/2 \text{ so } \sin(\phi) = -1$$

$$\Rightarrow \frac{m_1}{m_1 + m_2} v_{1I} = \omega A$$

$$\Rightarrow \boxed{v_{1I} = \frac{m_1 + m_2}{m_1} \omega A}$$

$$v_{1I} = \frac{10 + 15}{10}(0.35)(3)$$

$$\boxed{v_{1I} = 2.6 \text{ m/s}}$$

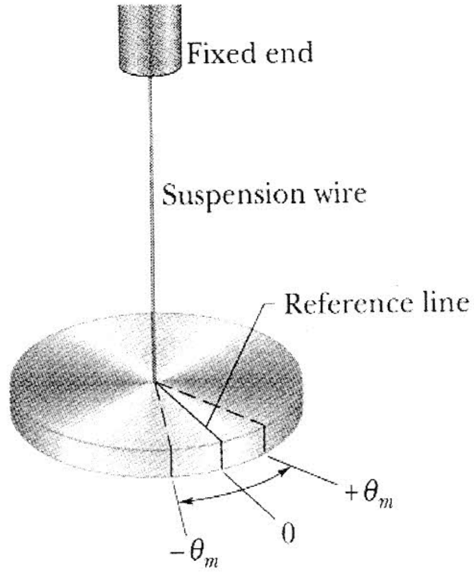
Oscillation – Set 2

The device in the picture below is known as a torsion pendulum. It is a flat disk attached to a length of stiff wire. When the wire is twisted, it responds by providing a torque on the disk, much the same way a spring provides a force when it is stretched. The torque provided by the wire is $T = -\kappa\theta$, where κ (greek letter kappa) is the torsion constant and theta is the angular displacement from equilibrium.

a) The moment of Inertia of the disk is $I = \frac{1}{2}MR^2$. Using the rotational version of Newton's Second Law, find the oscillator frequency of the torsion pendulum.

b) If a solid bar of length L, $I = \frac{1}{12}ML^2$, were suspended from the wire, what would the oscillator frequency be?

Let's solve the problem generally in terms of I, then plug in the different I for each object.



NSL for rotation

$$\Sigma T = I\alpha$$

$$-k\theta = I\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \boxed{\frac{d^2\theta}{dt^2} = -\frac{k}{I}\theta}$$ SHO!

This is the simple harmonic oscillator equation, except in θ instead of x.

$$\boxed{\omega = \left(\frac{k}{I}\right)^{1/2}}$$

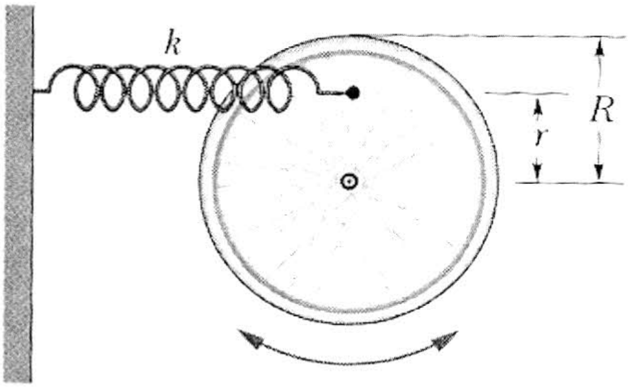
a) $\omega = \left[\frac{2k}{MR^2}\right]^{1/2}$ b) $\omega = \left[\frac{12k}{ML^2}\right]^{1/2}$

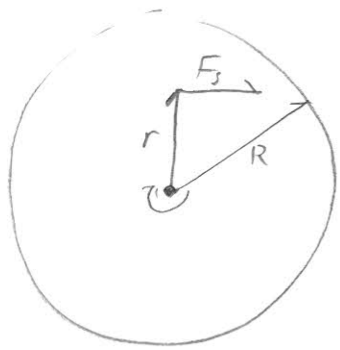
SAMPLE TEST 6

PHYS 111, FALL 2010, SECTION 1

A wheel is free to rotate about a fixed axle. A spring with a spring constant k is attached to one of its spokes at a distacne r from the axle, as shown in the picture. Assume that the wheel is a hoop of mass m and radius R (the spokes have negligible mass).

a) Using **Newton's Second Law**, find the angular frequency of small oscillations in terms of m, R, r and the spring constant k.
b) Using **Energy techniques**, find the angular frequency of small oscillations in terms of m, R, r and the spring constant k.
b) What is the angular frequency if r = R.
c) What is the angular frequency if r = 0.





a) $\sum \tau = I\alpha$

$F_s r = I\alpha$

$-kxr = I\alpha$, $x = r\theta$

$-kr\theta r = I\alpha$

$\alpha = -\boxed{\frac{kr^2}{I}}\theta \Rightarrow \omega = \left[\frac{kr^2}{I}\right]^{1/2}$, $I = mR^2$

$\Rightarrow \boxed{\omega = \left[\frac{kr^2}{mR^2}\right]^{1/2}}$

continued

Sample Test 6, P4 continued

b) $E_T = \frac{1}{2}kx^2 + \frac{1}{2}I\omega_r^2$ ← angular velocity!

$$\frac{dE_T}{dt} = \cancel{\frac{1}{2}}k\cancel{2}x\frac{dx}{dt} + \cancel{\frac{1}{2}}I\cancel{2}\omega_r\frac{d\omega_r}{dt} = 0$$

$$\Rightarrow kxv + I\omega_r\alpha = 0 \quad , \quad v = r\omega_r \Rightarrow \omega_r = \frac{v}{r}$$

$$\Rightarrow kx\cancel{v} + mR^2\frac{\cancel{v}}{r}\cdot\frac{a}{r} = 0 \quad a = r\alpha \Rightarrow \alpha = \frac{a}{r}$$

$$\Rightarrow kx + m\frac{R^2}{r^2}a = 0$$

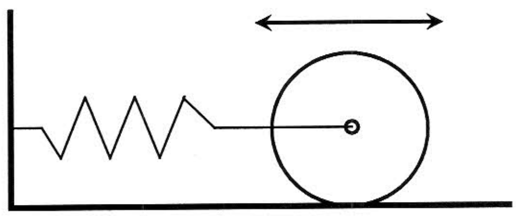
$$\Rightarrow a = -\boxed{\frac{k}{m}\frac{r^2}{R^2}}x \Rightarrow \boxed{\omega = \left[\frac{k}{m}\frac{r^2}{R^2}\right]^{1/2}}$$

c) if $R = r$, $\omega = \sqrt{\frac{k}{m}}$

d) if $r = 0$, $\underline{\omega = 0}$

Oscillation – Set 2

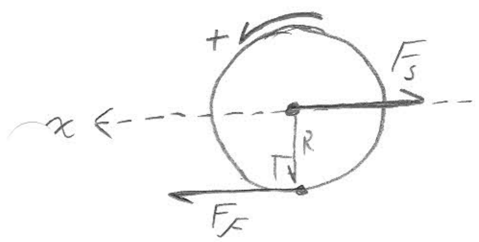
A solid cylinder of mass M is attached to a horizontal spring with force constant k. The cylinder can roll without slipping along the horizontal plane. When the system is displaced from the equilibrium position, it executes simple harmonic motion. Derive an expression for the period of the oscillations in terms of M, k, I and R.



This oscillator is rotating and translating, so we need Torque and translation.

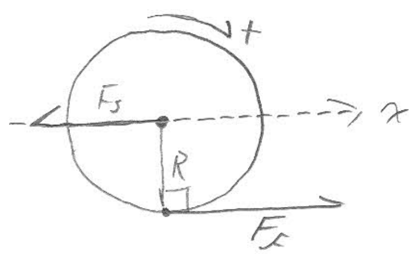
FBD → Pick the initial displacement from equilibrium as positive translation and rotation.

Displace left



Just to show that we can do it either way...

Displace right



① $\sum \tau = I\alpha \Rightarrow \boxed{-F_f R = I\alpha}$ ← same → $\sum \tau = I\alpha \Rightarrow \boxed{-F_f R = I\alpha}$

② $\sum F = ma \Rightarrow \boxed{F_f - F_s = ma}$ ← same → $\sum F = ma \Rightarrow \boxed{F_f - F_s = ma}$

solve ① for F_f and subst. into ②:

$$F_f = -\frac{I}{R}\alpha \xrightarrow{\text{into ②}} -\frac{I}{R}\alpha - \underbrace{kx}_{\text{spring}} = ma \Rightarrow -kx = ma + \underbrace{\frac{I}{R^2}a}_{\text{replaced } \alpha}$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\underbrace{\boxed{\frac{k}{I/R^2 + m}}}_{\omega^2} x$$

$$\underbrace{T}_{\text{Period}} = \frac{2\pi}{\omega} \Rightarrow \boxed{T = 2\pi\left[\frac{I/R^2 + m}{k}\right]^{1/2}}$$