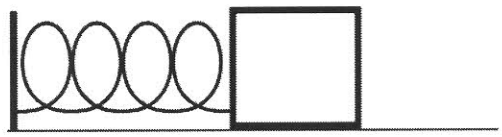
Oscillation – Set 4

The *Total Mechanical Energy* of a system, E_T, is the sum of the total ***Kinetic Energy*** and the total *Potential Energy*.

$$E_T = \sum K + \sum U$$

a) Write a general expression for the *Total Mechanical Energy* of a mass on a spring an an arbitrary point in time.



$E_T = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$

b) Assume that the oscillator above has an amplitude A and frequency ω.

When the system is at maximum compression what is its Kinetic Energy?

$K = 0$ because $v = 0$

When the system is at maximum compression what is its Potential Energy?

$U = \frac{1}{2} k A^2$ spring is compressed a distance A.

When the system is at equilibrium what is its Kinetic Energy?

$K = \frac{1}{2} m v_{max}^2 = \underline{\frac{1}{2} m A^2 \omega^2}$

When the system is at equilibrium what is its Potential Energy?

$U = 0$

An oscillating block-spring system has a mechanical energy of 1.00 J, an amplitude of of 10.0 cm, and a maximum speed of 1.20 m/s.

a) What is the spring constant?
b) What is the mass of the block?
c) What is the frequency of oscillation?

$$E_T = U + K \Rightarrow E_T = \tfrac{1}{2}kx^2 + \tfrac{1}{2}mv^2$$

a) Use the amplitude. When the spring is compressed, $K = 0$ and $x = A$

$$\text{So: } E_T = \tfrac{1}{2}kA^2 \Rightarrow \boxed{k = \frac{2E_T}{A^2}} \Rightarrow \boxed{k = \frac{(2)(1)J}{(0.1m)^2} = 200\ N/m}$$

b) Use the max speed.
Max speed is at equilibrium when $U = 0$

$$E_T = \tfrac{1}{2}mV_{max}^2 \Rightarrow \boxed{m = \frac{2E_T}{V_{max}^2}} \Rightarrow \boxed{m = \frac{2(1J)}{(1.2\,m/s)^2} = 1.4\ kg}$$

c) We can use V_{max} and A to find ω.

$$v(t) = -\omega A\cos(\omega t + \phi)$$

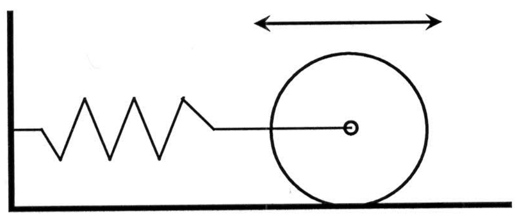
$$\Rightarrow V_{max} = \omega A \Rightarrow \boxed{\omega = \frac{V_{max}}{A}} \Rightarrow \boxed{\omega = \frac{1.2\,m/s}{0.1\,m} = 12\ rad/s}$$

Oscillation – Set ~~2~~ 3

Use **ENERGY** techniques to answer the following question.

A solid cylinder of mass M=2 kg and radius R=1.0 m is attached to a horizontal spring with spring constant k=100 N/m. The cylinder can roll without slipping along the horizontal plane. When the system is displaced from the equilibrium position, it executes simple harmonic motion.

a) Derive an expression for the period of the oscillations in terms of M, k, ~~and I~~.
b) If the cylinder has a translational velocity of v_0=5.0 m/s as it passes through equilibrium, find the phase constant, the amplitude, and the maximum acceleration of the system.



$$I = \tfrac{1}{2} m R^2 \qquad \omega = \frac{v}{R} \qquad \alpha = \frac{a}{R}$$

a) $$E_T = \tfrac{1}{2}kx^2 + \tfrac{1}{2}I\omega^2 + \tfrac{1}{2}mv^2$$

$$\frac{dE_T}{dt} = \frac{1}{2}k\,2x\frac{dx}{dt} + \frac{1}{2}I\,2\omega\frac{d\omega}{dt} + \frac{1}{2}m\,2v\frac{dv}{dt} = 0$$

$$\Rightarrow kxv + I\omega\alpha + mva = 0$$

$$\Rightarrow kx\cancel{v} + \tfrac{1}{2}m\cancel{R^2}\frac{\cancel{v}}{\cancel{R}}\frac{a}{\cancel{R}} + m\cancel{v}a = 0$$

$$\Rightarrow kx + \tfrac{1}{2}ma + ma = 0$$

$$\Rightarrow \tfrac{3}{2}ma = -kx$$

$$\Rightarrow a = -\frac{2}{3}\frac{k}{m}x \Rightarrow \frac{d^2x}{dt^2} = -\left[\frac{2}{3}\frac{k}{m}\right]x$$

$$\boxed{\omega = \left[\frac{2}{3}\frac{k}{m}\right]^{1/2}}$$

Oscillation set 3, P1 continued

b) $v_0 = 5.0$ m/s. Let's let $t=0$ then.

$\boxed{v(0) = 5.0 \text{ m/s}}$

At equilibrium, $x = 0$ so:

$\boxed{x(0) = 0}$

General Solution

$x(t) = A\cos(\omega t + \phi)$

$v(t) = -\omega A \sin(\omega t + \phi)$

$x(0) = A\cos(\phi) \Rightarrow 0 = A\cos(\phi) \Rightarrow \boxed{\phi = 0, \pi}$

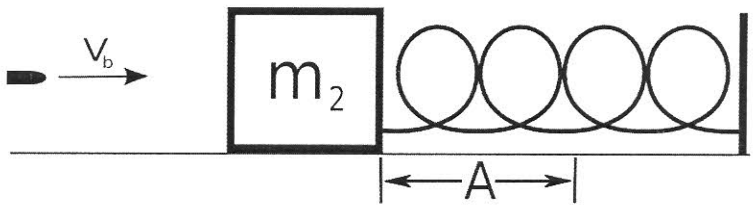
$v(0) = -\omega A \sin\phi \Rightarrow 5.0 = -\omega A \sin\phi$, let $\phi = \pi$

$\Rightarrow A = \frac{5.0}{\omega} = \sqrt{\frac{3}{2}\frac{m}{k}}\, 5.0$

$\boxed{A = \left(\frac{3}{\cancel{2}}\frac{\cancel{2}}{100}\right)^{1/2} \cdot 5.0 = 0.87 \text{ m}}$

A block of mass m_2 = 10 kg attached to a spring with spring constant k = 5 N/m is oscillating with an amplitude of A = 1.5 m horizontally on a frictionless surface. When the spring has reached its maximum extension to the left, it collides with a bullet with a mass m_1 = 5 g moving V_b = 200 m/s towards the right.

a) What is the period of the oscillator after the collision?
b) What is the amplitude of the oscillator after the collision?



a) Find the oscillator freq. with energy:

$$E_T = \tfrac{1}{2}kx^2 + \tfrac{1}{2}m_T v^2$$

$$\frac{dE_T}{dt} = \frac{1}{\cancel{2}}k\cancel{2}x\frac{dx}{dt} + \frac{1}{\cancel{2}}m_T\cancel{2}v\frac{dv}{dt}$$

$$\Rightarrow 0 = kxv + m_T v a$$

$$\Rightarrow 0 = kx + m_T a \Rightarrow a = -\overbrace{\left[\frac{k}{m_T}\right]}^{\omega^2} x$$

$$m_T = m_B + m_2$$

$$\omega = \left[\frac{k}{m_B + m_2}\right]^{1/2}$$

continued ↓

Oscillation Set 3, P3 continued

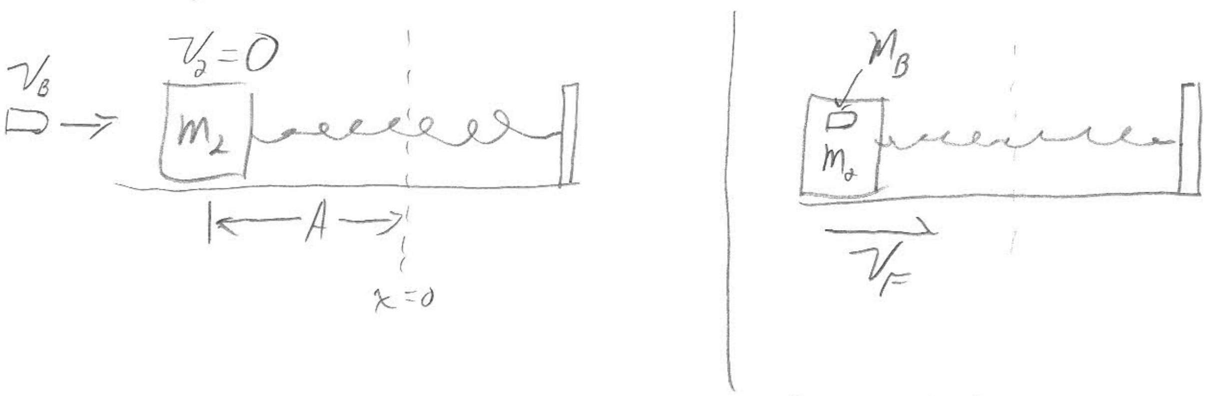
b) First, we have to determine the initial conditions.

Let's let the moment of collision be $t=0$.

at $t=0$, $\boxed{x(0)=A}$, the Amplitude of the old oscillator (pre-collision)

at $t=0$, $v(0)=v_F$, the post collision velocity of the Block/Bullet system.

To get v_F, conserve momentum



$$m_B v_B = (m_B + m_2) v_F$$

$$\Rightarrow \boxed{v_F = \frac{m_B}{m_B + m_2} v_B}$$

Now, the new oscillator will have a new amplitude A_n, and we can conserve energy to get A

continued

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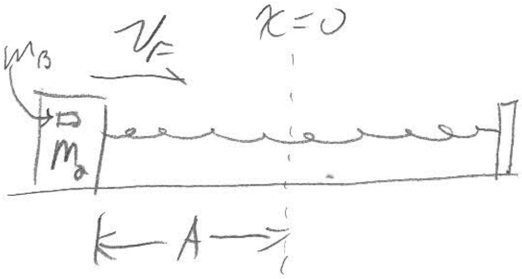
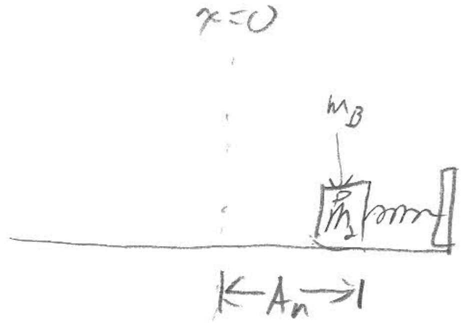
Oscillation Set 3 P3 continued





$U_I = \frac{1}{2}kA^2 \qquad K_I = \frac{1}{2}(m_2+m_B)v_F^2$

$U_F = \frac{1}{2}kA_n^2 \qquad K_F = 0$

$$\cancel{\tfrac{1}{2}}kA^2 + \cancel{\tfrac{1}{2}}(m_2+m_B)v_F^2 = \cancel{\tfrac{1}{2}}kA_n^2$$

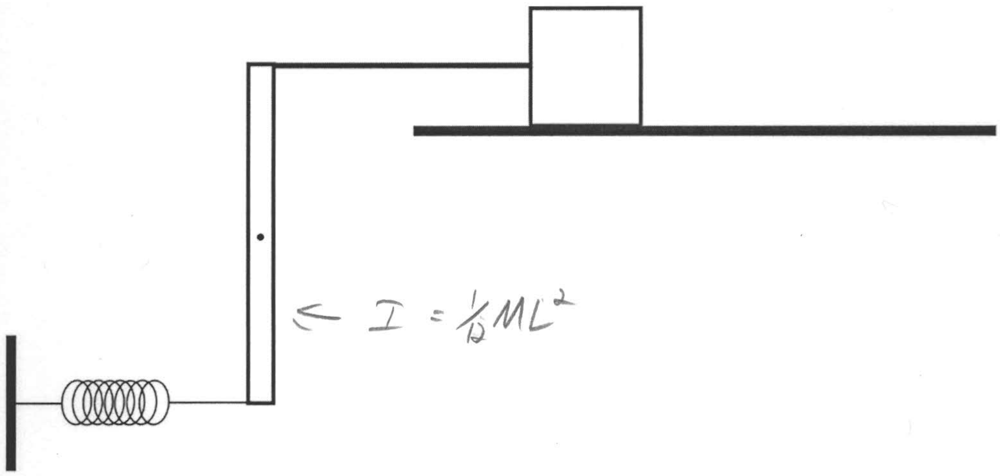
$$\Rightarrow \quad A_n = \left[A^2 + \frac{1}{k}(m_2+m_B)v_F^2\right]^{1/2}$$

$$\Rightarrow \quad A_n = \left[A^2 + \frac{1}{k}\cancel{(m_2+m_B)}\frac{m_B^2}{(m_2+m_B)^{\cancel{2}}}v_B^2\right]^{1/2}$$

$$\boxed{A_n = \left[A^2 + \frac{1}{k}\frac{m_B^2}{(m_2+m_B)}v_B^2\right]^{1/2}}$$

Use **ENERGY** techniques to answer the following question.

A block of mass *M* resting on a frictionless surface is attached to a stiff rod of negligible mass. The other end of the rod is attached to the top of a thin bar of length *l* mass *M* that is allowed to rotate about its center. The bottom of a bar is attached to a light spring of spring constant k. The spring is relaxed when the bar is vertical. Find the frequency of small oscillations.



$$E_T = \frac{1}{2}kx^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$$

$$\frac{dE_T}{dt} = \frac{1}{\cancel{2}}k\cancel{2}x\frac{dx}{dt} + \frac{1}{\cancel{2}}I\cancel{2}\omega\frac{d\omega}{dt} + \frac{1}{\cancel{2}}m\cancel{2}v\frac{dv}{dt} = 0$$

$$\Rightarrow kxv + I\omega\alpha + mva = 0 , \quad \omega = \frac{v}{L/2} , \; \alpha = \frac{a}{L/2}$$

$$\Rightarrow kx\cancel{v} + \frac{1}{12}m\cancel{L^2}\frac{\cancel{v}}{\cancel{L}/2}\cdot\frac{a}{\cancel{L}/2} + m\cancel{v}a = 0$$

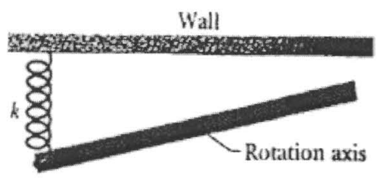
$$\Rightarrow kx + \frac{1\cancel{4}}{3\cancel{12}}ma + ma = 0$$

$$\Rightarrow \frac{4}{3}ma = -kx \Rightarrow a = -\frac{3}{4}\frac{k}{m}x \Rightarrow \boxed{\frac{d^2x}{dt^2} = -\left[\frac{3}{4}\frac{k}{m}\right]x}$$

$$\boxed{\omega = \left[\frac{3}{4}\frac{k}{m}\right]^{1/2}}$$

Oscillation – Set 3

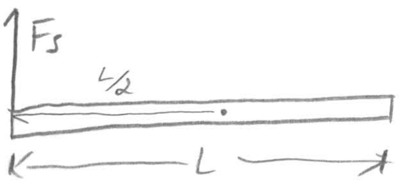
A long uniform rod of length L and mass m is free to rotate in a *horizontal* plane about a vertical axis through its center (the picture shows a *top* view). A spring with force constant k is connected horizontally between one end of the rod and a fixed wall. When the rod is in equilibrium, it is parallel to the wall.



What is the period of the *small* oscillations that result when the rod is rotated slightly and then released? $I_{cm} = \frac{1}{12}ML^2$ for the rod.

a) Use **Newton's Second Law** to find the oscillator frequency.
b) Use **Energy Techniques** to find the oscillator frequency.

a)



$$\sum \tau = I\alpha \Rightarrow \alpha = \frac{\sum \tau}{I}$$

$$\Rightarrow \alpha = \frac{\frac{L}{2}F_s}{\frac{1}{12}ML^2} \Rightarrow \alpha = \frac{-\frac{L}{2}kx}{\frac{1}{12}ML^2}, \quad x = \theta\frac{L}{2}$$

$$\Rightarrow \alpha = \frac{-\cancel{6}^{\,3}\,\theta\,\frac{\cancel{L}}{2}k}{M\cancel{L}} \Rightarrow \boxed{\alpha = -3\frac{k}{M}\theta}$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\left[\frac{3k}{M}\right]\theta \Rightarrow \boxed{\omega = \left[\frac{3k}{M}\right]^{1/2}}$$

b) $E_T = \frac{1}{2}kx^2 + \frac{1}{2}I\omega^2$

$$\frac{dE_T}{dt} = \frac{1}{2}k2x\frac{dx}{dt} + \frac{1}{2}I2\omega\frac{d\omega}{dt} = 0 \Rightarrow kxv + I\omega\alpha = 0$$

$$x = \frac{L}{2}\theta, \quad v = \frac{L}{2}\omega$$

$$k\frac{L}{2}\theta\frac{L}{2}\omega + \frac{1}{12}ML^2\omega\alpha = 0 \Rightarrow \frac{1}{12}ML^2\alpha = -\frac{1}{4}kL^2\theta \Rightarrow \boxed{\alpha = -3\frac{k}{M}\theta}$$

$$\boxed{\omega = \left[\frac{3k}{M}\right]^{1/2}}$$

Two particles are in simple harmonic motion in a straight line. They have the same amplitude and a period of 1.5 s but differ in phase by π/6 radians.

a) How far apart are they from one another (in terms of A) when the lagging particle is at its maximum position?

b) Are they moving in the same direction or opposite directions?

c) How far apart are they 0.5 seconds later?

d) Are they moving in the same or opposite directions then?

$A_1 = A_2 = A$, $T_1 = T_2 = T$, $\phi_1 = 0$, $\phi_2 = \pi/6$,

a) $x_1(t) = A_1 \cos(\omega_1 t + \phi_1)$

$x_2(t) = A_2 \cos(\omega_2 t + \phi_2)$

$$\boxed{\omega = \frac{2\pi}{T}}$$

what's t_0?

$x_1(t_0) = A\cos(\omega t_0)$ At max, $\underline{x = A}$

$A = A\cos(\omega t_0) \Rightarrow \cos(\omega t) = 1$

$\Rightarrow \omega t = 0 \Rightarrow \underline{\underline{t = 0}}$

$x_2(0) = A\cos(\phi_2) \Rightarrow x_2(0) = A\cos(\pi/6) = \frac{\sqrt{3}}{2} A$

$$\boxed{x_2 - x_1 = \left(\frac{\sqrt{3}}{2} - 1\right)A}$$

b) $v_1(0) = -\omega A \sin(0) = 0$

hmm... neither...
trick question :)

c) at $t=0.5$

$x_1(0.5) = A\cos(\omega \cdot 0.5)$, $T = \frac{3}{2} \Rightarrow \frac{2\pi}{3/2} = \frac{4}{3}\pi = \omega$

$x_1(0.5) = A\cos(\frac{4}{3}\pi \frac{1}{2})$ $\quad \omega = \frac{4}{3}\pi$

$x_1(0.5) = A\cos(\frac{2}{3}\pi)$

$$\underline{x_1(0.5) = -\frac{1}{2}A}$$

$x_2(0.5) = A\cos(\frac{4}{3}\pi \cdot \frac{1}{2} + \frac{\pi}{6})$

$= A\cos(\frac{5}{6}\pi)$

$= -\frac{\sqrt{3}}{2}A$

$$\boxed{x_2 - x_1 = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)A}$$

d) $v_1(0.5) = -\omega A \sin(\frac{2}{3}\pi) = -\omega A \frac{\sqrt{3}}{2}$

$v_2(0.5) = -\omega A \sin(\frac{5}{6}\pi) = -\omega A \frac{1}{2}$

same sign, <u>same</u> direction