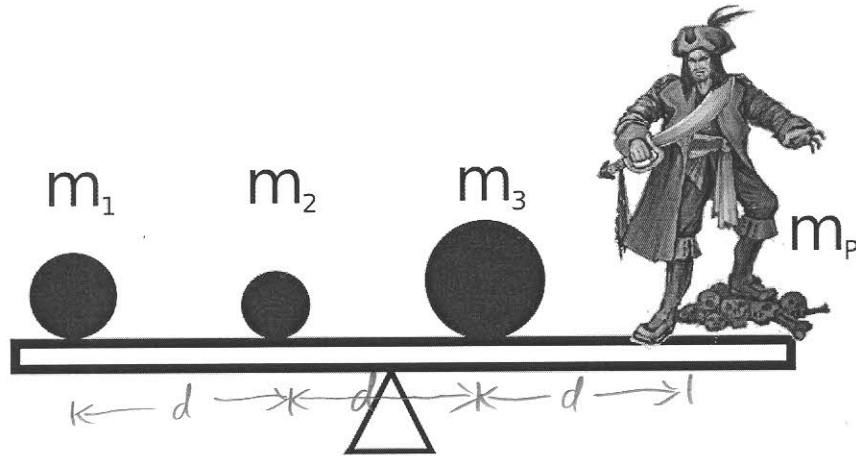


Systems of Particles – Set 1

For reasons that nobody can explain, the Evil Pirate wants to stand on one end of a Plank of Negligible Mass (it's presumably used for walking) and be perfectly balanced by balls from the *Horrible Pendulum of Doom*. He's a little bit OCD, so he insists that all of the objects on the plank must be separated by the same distance d .



You can avoid walking the Plank of Negligible Mass if you can tell him where to put his fulcrum so that he can stand happily balanced and calm.

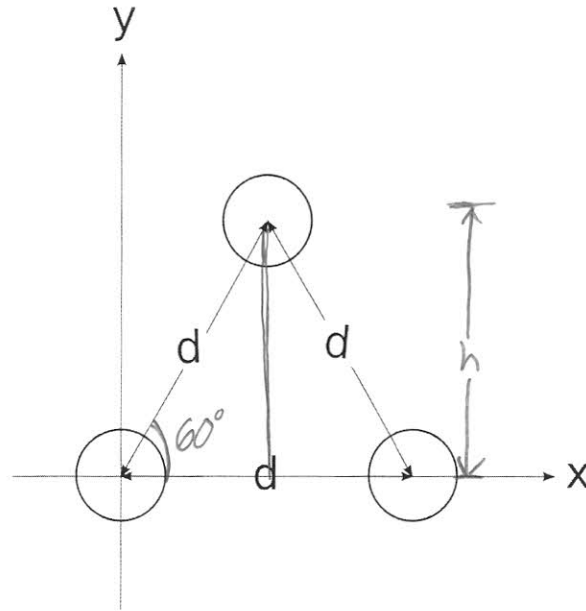
HINT: Put the fulcrum at the center of mass.

I'll put m_1 at $x=0$ and calculate the center of mass using the Mass Weighted Average Position.

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1(0) + m_2(d) + m_3(2d) + m_p(3d)}{m_1 + m_2 + m_3 + m_p}$$

$$x_{cm} = d \frac{m_2 + 2m_3 + 3m_p}{m_1 + m_2 + m_3 + m_p}$$

Three balls of equal mass form an equilateral triangle. Find the coordinates of the center of mass of the system.



2D problem, calculate x_{cm} and y_{cm} separately.

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i} = \frac{0 \cdot m + \frac{d}{2}m + dm}{3m} = \frac{dm}{3m} \left(\frac{1}{2} + 1 \right) = \frac{d}{3} \cdot \frac{3}{2} = \boxed{\frac{d}{2}}$$

$$y_{cm} = \frac{\sum m_i y_i}{\sum m_i} = \frac{0 \cdot m + 0 \cdot m + hm}{3m}$$

$h = d \sin 60$
 $\Rightarrow h = d \frac{\sqrt{3}}{2}$

$$y_{cm} = \frac{d \frac{\sqrt{3}}{2} m}{3m} = d \frac{\sqrt{3}}{2 \cdot 3} = \boxed{\frac{d}{2\sqrt{3}}}$$

Systems of Particles – Set 1

We are going to calculate the location of the center of mass of a thin uniform rod of mass M and length L . The integral form of the center of mass equation is:

$$l_{cm} = \frac{\int_{l_0}^L l dm}{\int_{l_0}^L dm}$$

The variable of integration is l , but the differential is dm . We need to rewrite dm in terms of l . In other words, we need to perform a *Change of Variables*. The following steps will walk us through it.

- a. You're in the hardware store and you notice that chain is on sale for \$1.00 per pound. The chain you want is 3 pounds per foot and you want 10 feet. How much do you have to spend?

One to one dollar to pound correspondance, so lb and \$ are the same

$$\left(3 \frac{\text{lb}}{\text{ft}}\right) \cdot 10 \text{ft} = \underline{30 \text{ lb}} \quad \text{or} \quad \underline{\$30}$$

- b. You see that your neighbor is walking out the door with 50 feet of chain. He says gleefully that he paid only 25 dollars for it. What is the *linear density* (mass per unit length) of his chain ?

$$\$25 = 25 \text{ lb}$$

$$\lambda = \frac{25 \text{ lb}}{50 \text{ ft}} = \left[\frac{1 \text{ lb}}{2 \text{ ft}} \right]$$

linear density

- c. What would you pay for 30 feet of the chain that your neighbor bought?

linear density times length gives mass.

$$\frac{1 \text{ lb}}{2 \text{ ft}} \cdot 30 \text{ft} = 15 \text{ lb} \quad \text{or} \quad \underline{\$15}$$

- d. We generally assign linear density the variable λ . What is the linear density (λ) of a uniform rod of mass M and length L ? (refer to step b for assistance)

Linear density is mass over distance

$$\text{So: } \left[\lambda = \frac{M}{L} \right]$$

d. Now, write an equation for the mass m of a piece of this rod in terms of the total mass M , the total length L , and the length of the piece l . (refer to step c for assistance).

linear density times length equals mass ...

$$\underline{m = \lambda l} \Rightarrow \boxed{m = \frac{M}{L} l}$$

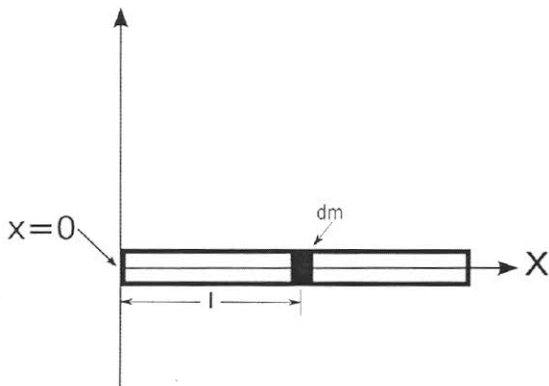
e. Calculate the derivative $\frac{dm}{dl}$ of the equation above. Now, what is dm in terms of M , L , and dl ?

$$\frac{dm}{dl} = \frac{d}{dl} \lambda l = \lambda$$

This is REALLY important

$$\frac{dm}{dl} = \lambda \Rightarrow \boxed{dm = \lambda dl} \Rightarrow \boxed{dm = \frac{M}{L} dl}$$

f. Taking the lower integral first, use the picture below and what we discovered in part e to calculate the total mass. Hopefully you get M .



$$\int_0^L dm = \int_0^L \frac{M}{L} dl = \frac{M}{L} \int_0^L dl$$

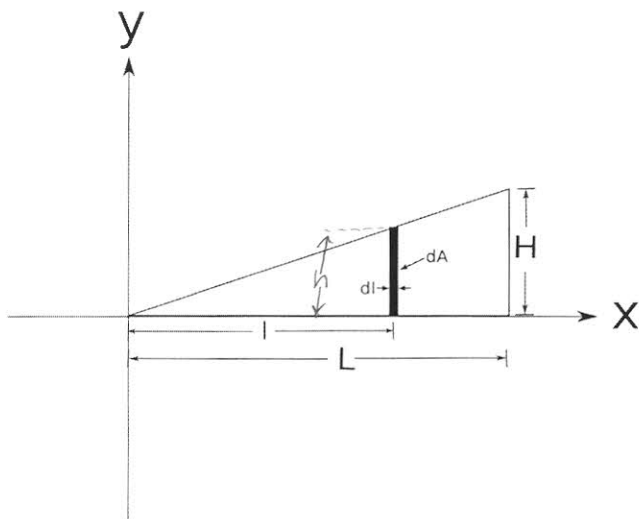
$$= \frac{M}{L} L = \boxed{M}$$

g. Now calculate the the position of the center of mass. Hopefully, you get $L/2$.

$$l_{cm} = \frac{\int l dm}{\int dm} = \frac{1}{M} \int_0^L l \frac{M}{L} dl = \frac{1}{L} \int_0^L l dl = \frac{1}{L} \left(\frac{1}{2} l^2 \right) \Big|_0^L = \frac{1}{2} L$$

$$\boxed{l_{cm} = \frac{L}{2}}$$

Calculate the center of mass of a triangular chunk of aluminum of mass M , length l , and height h .



a) Write an expression for dm in terms of dl similarly to what we did for the uniform rod.

Now, we consider a slice of the triangle with mass dm and area dA .

The slice has height h and width dl so: $dA = h dl$

Linear density λ times length gives mass so...

Surface density (σ) times area gives mass

$$dm = \sigma dA \Rightarrow dm = \sigma h dl$$

σ has units of mass over area so:

$$\sigma = \frac{M}{\frac{1}{2}LH} = \frac{2M}{LH} \Rightarrow dm = \frac{2M}{LH} h dl$$

continued

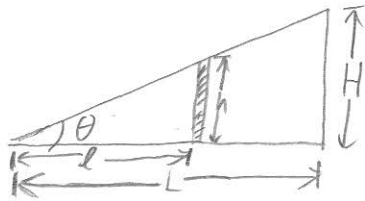


Systems Set 1 - P5 continued

b) calculate x_{cm}

$$x_{cm} = \frac{1}{M} \int_0^L l dm = \frac{1}{M} \int_0^L l \frac{2M}{LH} h dl = \frac{2}{LH} \int_0^L l h dl$$

h depends on l so we have to rewrite it:



$$\tan \theta = \frac{H}{L}, \quad \tan \theta = \frac{h}{l}$$

$$\text{so: } \frac{H}{L} = \frac{h}{l} \Rightarrow \boxed{h = l \frac{H}{L}}$$

$$x_{cm} = \frac{2}{LH} \int_0^L l \cdot l \frac{H}{L} dl = \frac{2}{L^2} \int_0^L l^2 dl = \frac{2}{L^2} \left(\frac{1}{3} l^3 \right) \Big|_0^L = \frac{2}{L^2} \cdot \frac{1}{3} L^3$$

$$\boxed{x_{cm} = \frac{2}{3} L}$$

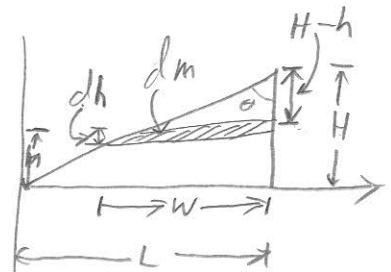
c) calculate y_{cm}

$$y_{cm} = \frac{1}{M} \int_0^H h dm = \frac{1}{M} \int_0^H h \frac{2M}{HL} w dh$$

$$= \frac{2}{HL} \int_0^H h \frac{1}{H} (H-h) dh$$

$$= \frac{2}{H^2} \int_0^H (hH - h^2) dh = \frac{2}{H^2} \left(\frac{1}{2} H^3 - \frac{1}{3} H^3 \right)$$

$$= H \left(1 - \frac{2}{3} \right) = \boxed{\frac{1}{3} H}$$



$$dm = \sigma dA$$

$$\boxed{dm = \frac{2M}{HL} w dh}$$

$$\text{and } \frac{w}{H-h} = \frac{L}{H}$$

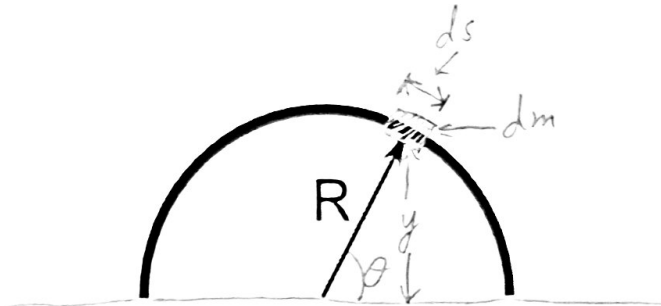
$$\Rightarrow \boxed{w = \frac{L}{H} (H-h)}$$

Systems of Particles – Set 1

6

A thin rod of mass M has been bent into a semi-circle with radius R .

Find the coordinates of its center of mass.



$$r_{cm,y} = \frac{1}{M} \int y dm, \quad \text{linear so: } dm = \lambda ds \quad \text{and} \quad \lambda = \frac{M}{\pi R}$$

$$\Rightarrow r_{cm,y} = \frac{1}{M} \int y \frac{M}{\pi R} ds$$

change to θ :

$$y = R \sin \theta, \quad ds = R d\theta$$

$$\Rightarrow r_{cm,y} = \frac{1}{M} \int_0^{\pi} R \sin \theta \frac{M}{\pi R} R d\theta$$

$$= \frac{R}{\pi} \int_0^{\pi} \sin \theta d\theta = \frac{R}{\pi} \left(-\cos \theta \right) \Big|_0^{\pi}$$

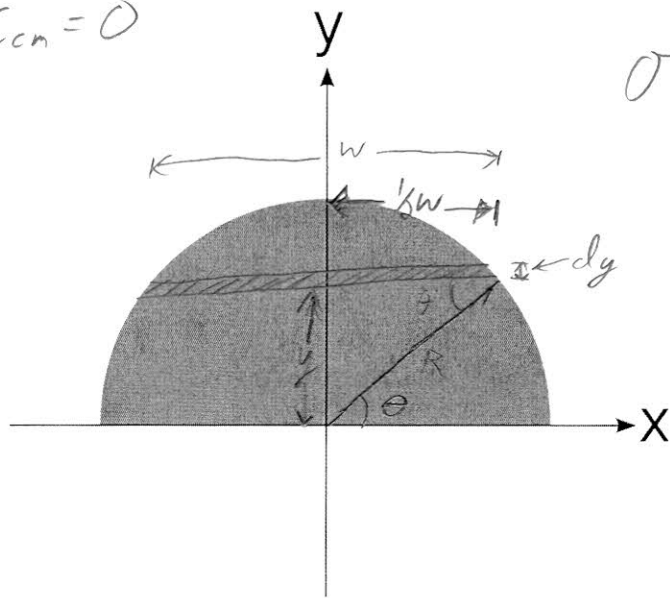
$$\Rightarrow \boxed{r_{cm,y} = \frac{2R}{\pi}}$$

Systems of Particles – Set 1

6 5

A flat piece of aluminum is cut into a semi-circle of radius R . Calculate the coordinates of the center of mass.

By symmetry, $x_{cm} = 0$



$$\sigma = \frac{M}{A} = \frac{M}{\frac{1}{2}\pi R^2}$$

$$y_{cm} = \frac{1}{M} \int y dm, \quad dm = \sigma w dy$$

What's w ?

Well, according to pythagoras, $y^2 + (\frac{1}{2}w)^2 = R^2$

$$\Rightarrow \frac{1}{4}w^2 = R^2 - y^2$$

$$\Rightarrow w = 2\sqrt{R^2 - y^2}$$

Okay

$$dm = \frac{2M}{\pi R^2} 2\sqrt{R^2 - y^2} dy$$

continued



Systems of Particles, Set 1, ^{P5} ~~P6~~ continued

$$y_{cm} = \frac{1}{M} \int_0^R y \frac{4M}{\pi R^2} \sqrt{R^2 - y^2} dy = \frac{4}{\pi R^2} \int_0^R y \sqrt{R^2 - y^2} dy$$

hmmm... how about a substitution?

like $u = R^2 - y^2$, then $\frac{du}{dy} = -2y \Rightarrow \underline{dy = -\frac{du}{2y}}$

when: $y = R, u = 0$

$y = 0, u = R^2$

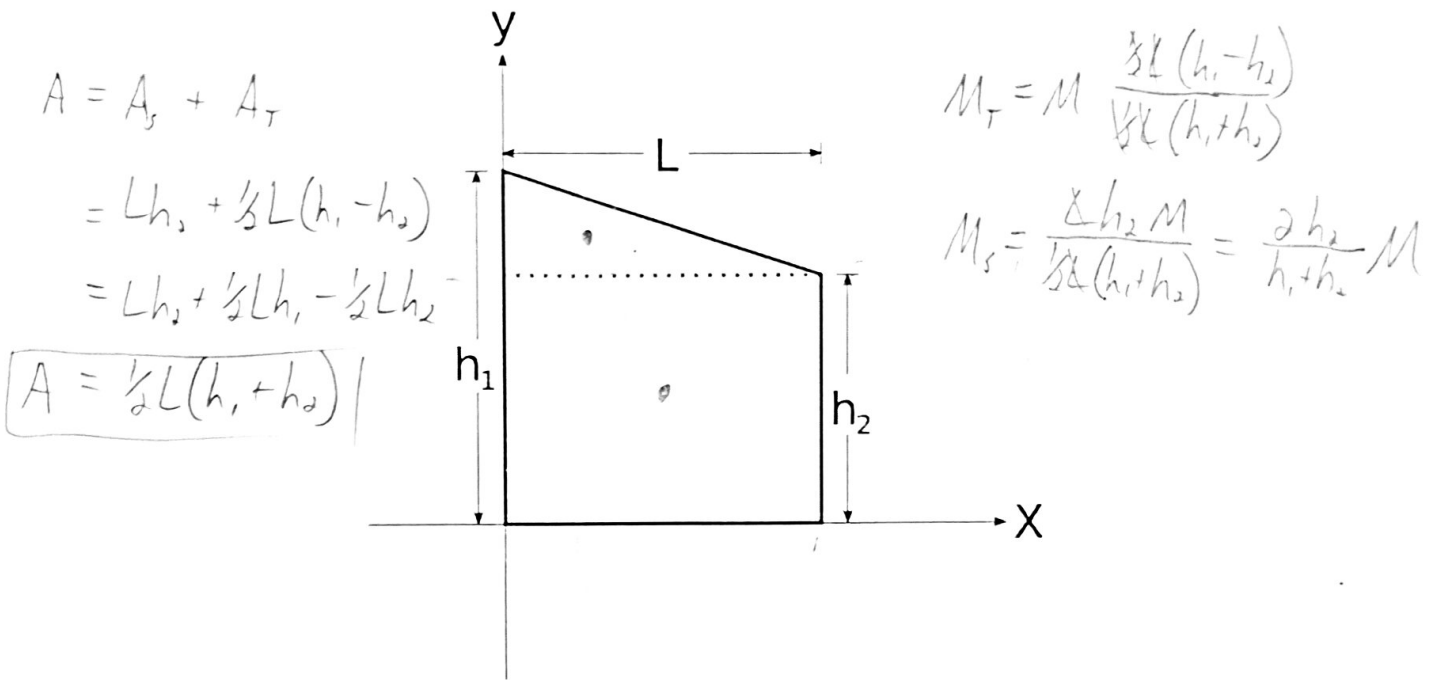
$$\text{So } y_{cm} = -\frac{2M}{\pi R^2} \int_{R^2}^0 y (u)^{\frac{1}{2}} \frac{du}{2y} = \frac{2}{\pi R^2} \int_0^{R^2} u^{\frac{1}{2}} du$$

$$= \frac{2}{\pi R^2} \left(\frac{2}{3} u^{\frac{3}{2}} \right) \Big|_0^{R^2} = \frac{4}{3} \frac{1}{\pi R^2} (R^2)^{\frac{3}{2}} = \frac{4}{3} \frac{R^3}{\pi R^2}$$

$$\boxed{y_{cm} = \frac{4}{3\pi} R}$$

Systems of Particles – Set 1

The two dimensional shape below has a mass M and has been cut out of uniform sheet of aluminum. Calculate the x coordinate of the center of mass. Use symmetry arguments to calculate the y coordinate of the center of mass.



We can treat the object as a square attached to a triangle. Then treat each object as a point concentrated at its own center of mass.

For the square: $\vec{r}_{cns} = \frac{1}{2}L \hat{x} + \frac{1}{2}h_2 \hat{y}$

For the triangle; $\vec{r}_{cmt} = \frac{1}{3}L \hat{x} + h_2 + \frac{1}{3}(h_1 - h_2) \hat{y}$
 (Stealing triangle result from earlier problem.)

Set 1, PS - continued

So: $r_{cmx} = \frac{1}{M} \sum m_i r_{xi}$ ← two terms in sum, one for triangle, one for square.

$$r_{cmx} = \frac{1}{M} \cdot (M_T r_{Tx} + M_S r_{Sx})$$

$$= \frac{1}{M} \left(M \frac{h_1 - h_2}{h_1 + h_2} \cdot \frac{1}{3} L + M \frac{2h_2}{h_1 + h_2} \cdot \left(\frac{1}{2} L \right) \right)$$

$$= \frac{1}{h_1 + h_2} \left(\frac{1}{3} h_1 L - \frac{1}{3} h_2 L + h_2 L \right)$$

$$\boxed{r_{cmx} = \frac{\frac{1}{3} h_1 + \frac{2}{3} h_2}{h_1 + h_2} L}$$

and: $r_{cmg} = \frac{1}{M} \sum m_i r_{iy}$

$$= \frac{1}{M} \left(M \frac{h_1 - h_2}{h_1 + h_2} \cdot \left(h_2 + \frac{1}{3} (h_1 - h_2) \right) + M \frac{2h_2}{h_1 + h_2} \cdot \frac{1}{2} h_2 \right)$$

$$= \frac{1}{h_1 + h_2} \left((h_1 - h_2) \left(\frac{1}{3} h_1 + \frac{2}{3} h_2 \right) + h_2^2 \right)$$

$$= \frac{1}{h_1 + h_2} \left(\frac{1}{3} h_1^2 + \frac{2}{3} h_1 h_2 - \frac{1}{3} h_1 h_2 - \frac{2}{3} h_2^2 + h_2^2 \right)$$

$$\boxed{r_{cmg} = \frac{1}{h_1 + h_2} \left(\frac{1}{3} h_1^2 + \frac{1}{3} h_1 h_2 + \frac{1}{3} h_2^2 \right)}$$