Name: ______________________ **Problems Solved __/ 8**

A resistor with resistance R is wired in parallel with a capacitor with capacitance C. The capacitor is initially charged up to a voltage ΔV_0. Because the capacitor puts a voltage across the resistor, current will flow through the resistor.

Underline the correct response corresponding to how each quantity changes as a function of time.

The charge on the positive capacitor plate:
Increases Remains Constant <u>Decreases</u>

The E field between the capacitor plates:
Increases Remains Constant <u>Decreases</u>

The potential difference between the capacitor plates:
Increases Remains Constant <u>Decreases</u>

The total energy stored in the capacitor:
Increases Remains Constant <u>Decreases</u>

The potential difference across the resistor:
Increases Remains Constant <u>Decreases</u>

The current through the Resistor:
Increases Remains Constant <u>Decreases</u>

The power dissipated by the Resistor:
Increases Remains Constant <u>Decreases</u>

Write three expressions for the the total energy stored in the circuit (think back to when we studied capacitors).

$U = \frac{1}{2}CV^2$, $U = \frac{1}{2}\frac{Q^2}{C}$, $U = \frac{1}{2}QV$

Write an expression that relates the voltage across the resistor to the current through it (we derived it recently).

$V = IR \rightarrow$ Ohm's Law

Write an expression for the power dissipated through the resistor given the current and voltage.

$P = VI$

Because *Power* is the time rate of change of energy ($P = \frac{dU}{dt}$), the power equation written above suggests a *differential equation.* Rewrite the power equation in terms of the rate of change of energy to form a differential equation.

$$\frac{dU}{dt} = VI$$

Your differential equation should have the variables U, V, and I. Let's rewrite the equation in terms of the voltage across the capacitor, V. Using relationships that you wrote down on the previous page, perform a change variables from U and I to V.

$$U = \tfrac{1}{2}CV^2 \Rightarrow \frac{dU}{dt} = \tfrac{1}{2}C\,2V\frac{dV}{dt} \Rightarrow \boxed{\frac{dU}{dt} = CV\frac{dV}{dt}}$$

$$\boxed{I = \frac{V}{R}}$$

$$\Rightarrow \frac{dU}{dt} = VI \Rightarrow C\cancel{V}\frac{dV}{dt} = \frac{V^{\cancel{2}}}{R} \Rightarrow \frac{dV}{dt} = \frac{V}{RC}$$

But! $\frac{dV}{dt}$ is negative in our case! $\Rightarrow \boxed{\frac{dV}{dt} = -\frac{V}{RC}}$

Now, separate and integrate. What are the proper limits of integration?

$$\int_{V_0}^{V} \frac{1}{V}\,dV = -\int_{0}^{t} \frac{1}{RC}\,dt \Rightarrow \ln\left(\frac{V}{V_0}\right) = -\frac{t}{RC}$$

$$\Rightarrow \boxed{V = V_0 e^{-t/RC}}$$

– or –

$$V = V_0 e^{-t/\tau} \quad \text{where } \tau = RC$$

A resistor with resistance R is wired in series with a capacitor with capacitance C and connected to a battery with voltage V_b. The capacitor is initially uncharged.

Underline the correct response corresponding to how each quantity changes as a function of time.

The charge on the positive capacitor plate:
<u>Increases</u> Remains Constant Decreases

The E field between the capacitor plates:
<u>Increases</u> Remains Constant Decreases

The potential difference between the capacitor plates:
<u>Increases</u> Remains Constant Decreases

The total energy stored in the capacitor:
<u>Increases</u> Remains Constant Decreases

The potential difference across the resistor:
Increases Remains Constant <u>Decreases</u>

The current through the Resistor:
Increases Remains Constant <u>Decreases</u>

The power dissipated by the Resistor:
Increases Remains Constant <u>Decreases</u>

Begin with the same differential equation as before and rewrite the equation in terms of the *voltage on the resistor*. We should get the same equation as before. Separate and integrate.

$$\frac{dV}{dt} = -\frac{V}{RC} \Rightarrow \int_{V_b}^{V_R} \frac{1}{V}\,dV = -\int_0^t \frac{1}{RC}\,dt$$

$$V_R = V_b e^{-t/RC}$$

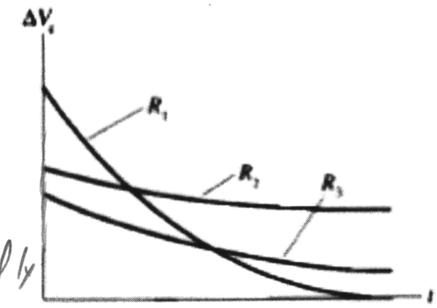
Now, given that we have an expression for the voltage on the resistor, find an expression for the voltage on the capacitor.

ah... $V_R + V_C = V_b \Rightarrow V_C = V_b - V_R$

$$\Rightarrow V_C = V_b - V_b e^{-t/RC}$$

$$\Rightarrow \boxed{V_C = V_b\left(1 - e^{-t/RC}\right)}$$

The graph shows the voltage vs. time for a capacitor as it is discharged (separately) through three different resistors. Rank, in order from largest to smallest, the values of the resistances R_1 to R_3. Explain your reasoning.



$$V_c = V_0 e^{-t/RC}$$

For a given time t, as R↑ $e^{-t/RC}$↓ so V_c derease more rapidly

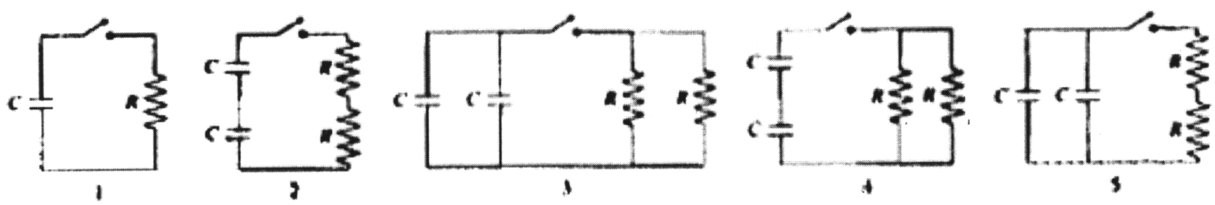
⇒ Faster decay means smaller time constant.

⇒ The Starting ΔV is ΔV(t=0) just refers to the initial capacitor charge

⇒ The slope at a given time t depends on RC (inversely)

$$\boxed{R_1 < R_3 < R_2}$$

The capacitors in each circuit are discharged when the switch closes at t=0. Rank, in order from largest to smallest, the time constants τ_1 to τ_5 with wich each circuit will discharge. Explain your answer.



Find the "effective" time constant for each circuit.

1) RC

2) $(R_1 + R_2)\left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} = 2R \cdot \frac{1}{2}C = RC$

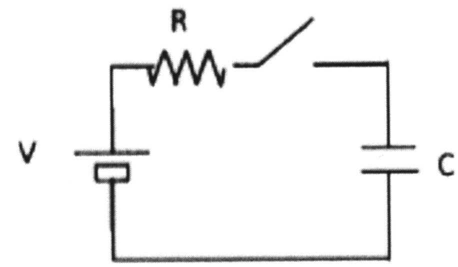
3) $\left(\frac{1}{R} + \frac{1}{R}\right)^{-1}(C + C) = \frac{1}{2}R + 2C = RC$

4) $\left(\frac{1}{C} + \frac{1}{C}\right)^{-1}\left(\frac{1}{R} + \frac{1}{R}\right)^{-1} = \frac{1}{2}C + \frac{1}{2}R = \frac{1}{4}RC$

5) $(C + C)(R + R) = 2C \cdot 2R = 4RC$

$$\boxed{\tau_5 > \tau_1 = \tau_2 = \tau_3 > \tau_4}$$

A 6V battery is used to charge a 2μf capacitor through a 100Ω resistor.



a) Find the initial current just after the switch is closed. At this moment, what is the potential difference across the resistor?
b) What is the final current after the switch has been closed for a long time?
c) What is the final potential difference across the capacitor and how much final charge does it carry?
d) How much time is required for the capacitor's charge to reach 90% of its final value?

a) $V_R = V_b e^{-t/RC}$, $V = IR$

at $t=0$, $V_R = V_B \Rightarrow \boxed{I = \frac{V_b}{R}} \Rightarrow I = \frac{6V}{100\Omega} = \boxed{60\,mA}$

b) as $t \to \infty$, $e^{-t/RC} \to 0$

so $V_R \to 0$ and $\boxed{I \to 0}$

c) $V_c = V_b(1 - e^{-t/RC}) \Rightarrow$ as $t \to \infty$, $V_c \to V_b$

So, $\boxed{V_c = V_b}$

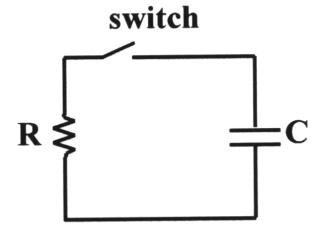
and $C = \frac{Q}{V} \Rightarrow \boxed{Q = CV_b} \Rightarrow \boxed{Q = (2\times10^{-6}F)(6V) = 12\mu C}$

d) It will reach 90% when the voltage gets to 90% of the battery voltage:

$\frac{V_c}{V_b} = 0.9 \Rightarrow \frac{\cancel{V_b}(1-e^{-t/RC})}{\cancel{V_b}} = 0.9 \Rightarrow 1 - 0.9 = e^{-t/RC}$

$\Rightarrow \ln(0.1) = -t/RC \Rightarrow \boxed{t = RC\ln(10)}$

$\Rightarrow \boxed{t = (100)(2\times10^{-6})\ln(10) = 461\,\mu s}$



A capacitor is being discharged through a resistor R. The capacitor's initial charge is Q_0.

a) After how many time constants is the charge on the capacitor one fourth of its initial value?

b) The energy stored in the capacitor decreases with time as it discharges. After how many time constants is the stored energy one fourth of its initial value?

c) After how many time constants is the current in the RC circuit one half of its initial value?

d) A 10 μf capacitor is charged by a 10V battery through a resistance R. The capacitor reaches a potential difference of 4.0V in 3.0 seconds after charging begins. Find R.

a) Charge goes as Voltage: $Q = CV \Rightarrow Q/\cancel{C} = \frac{Q_0}{\cancel{C}} e^{-t/RC}$

$$\Rightarrow \frac{Q}{Q_0} = e^{-t/RC} \Rightarrow \ln\left(\frac{Q_0}{Q}\right) = \boxed{\frac{t}{RC}} \rightarrow \boxed{\text{\# of time constants}}$$

$$\boxed{\ln\left(\frac{a}{b}\right) = -\ln\left(\frac{b}{a}\right)}$$

and we want $\frac{Q}{Q_0} = \frac{1}{4} \Rightarrow \frac{t}{RC} = \ln(4) \Rightarrow \boxed{t/RC = 1.39}$

b) $U = \frac{1}{2}CV^2$ Energy goes as Voltage squared

$$\Rightarrow V = \left(2\frac{U}{C}\right)^{1/2} \Rightarrow \left[2\frac{U}{\cancel{C}}\right]^{1/2} = \left[2\frac{U_0}{\cancel{C}}\right]^{1/2} e^{-t/RC}$$

$$\Rightarrow \left[\frac{U}{U_0}\right]^{1/2} = e^{-t/RC} \Rightarrow \frac{U}{U_0} = e^{-2t/RC}$$

$$\Rightarrow \ln\left(\frac{U_0}{U}\right) = \frac{2t}{RC} \Rightarrow \boxed{\frac{1}{2}\ln\left(\frac{U_0}{U}\right) = t/RC}$$

and we want $\frac{U}{U_0} = \frac{1}{4} \Rightarrow \boxed{t/RC = \frac{1}{2}\ln(4) = 0.69 \text{ time const.}}$

continued ↓

Circuits Set 2, #6 continued

c) on the resistor: $V_R = V_b e^{-t/RC}$, and $V_R = I_R R$

$$\Rightarrow I_R R = I_0 R e^{-t/RC}$$

$$\Rightarrow \frac{I}{I_0} = e^{-t/RC} \Rightarrow \boxed{\ln\left(\frac{I_0}{I}\right) = t/RC}$$

$$\Rightarrow \ln(2) = \boxed{t/RC = 0.69 \text{ time constants}}$$

d) Given

$C = 10\mu F$

$V_b = 10v$

$V(t_f) = 4.0V$

$t_f = 3.0s$

$$V_c = V_b(1 - e^{-t/RC})$$

$$\Rightarrow 1 - \frac{V_c}{V_b} = e^{-t/RC}$$

$$\Rightarrow \frac{V_b - V_c}{V_b} = e^{-t/RC}$$

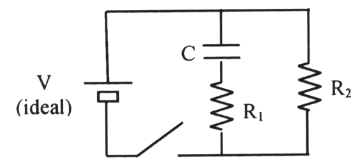
$$\Rightarrow \ln\left(\frac{V_b}{V_b - V_c}\right) = \frac{t}{RC}$$

$$\Rightarrow \boxed{R = \frac{t}{C}\left[\ln\left(\frac{V_b}{V_b - V_c}\right)\right]^{-1}}$$

$$\Rightarrow R = \frac{3.0s}{10\times10^{-6}F}\left[\ln\left(\frac{10V}{10V - 4V}\right)\right]^{-1}$$

$$\boxed{R = 590\,k\Omega}$$

In the circuit shown at right, the capacitor is originally uncharged with the switch open. At t = 0 the switch is closed. Write your answers in algebraic form until part (c).

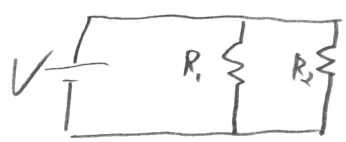


a) What is the current supplied by the battery just after the switch is closed?

b) What is the current a long time after the switch is closed?

c) After a long time, the switch is opened. How long does it take for the charge on the capacitor to decrease to 10 percent of its value at t = t' if $R_1 = R_2 = 5\ \mathrm{k\Omega}$ and $C = 1.0\ \mu\mathrm{F}$?

a) At $t=0$, $\Delta V_c = 0$, so the equivalent circuit is:



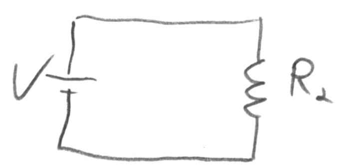
$$\Rightarrow R_{eff} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} = \frac{R_1 R_2}{R_1 + R_2}$$

$$\Rightarrow I_0 = \frac{V_b}{R_{eff}} \Rightarrow \boxed{I_0 = V_b \cdot \frac{R_1 + R_2}{R_1 R_2}}$$

b) As $t \to \infty$ $\Delta V_c \to V_b \Rightarrow \Delta V_{R_1} \to 0 \Rightarrow I_{R_1} \to 0$

so all current flows through R_2.

equivalent circuit:



$$\Rightarrow \boxed{I_f = \frac{V_b}{R_2}}$$

c) After a long time, $V_c = V_b$. Want $\frac{V_c}{V_b} = 0.1$

With the switch open;

$$V_c = V_b e^{-t/RC}$$

$$\Rightarrow t/RC = \ln\left(\frac{V_b}{V_c}\right) \Rightarrow t = (R_1 + R_2)(C)\ln(10)$$

$$\boxed{t = (10 \times 10^3)(1 \times 10^{-6})\ln(10) = 23\ \mathrm{ms}}$$

Show that only half the total energy drawn from a battery in charging an *RC* circuit ends up stored in the capacitor. *Hint:* What happens to the rest? You will need to integrate!

Power delivered by the battery is: $P = VI$

or: $\frac{dU}{dt} = VI$, Battery voltage is constant

But $I = I_0 e^{-t/RC}$

so: $\frac{dU}{dt} = VI_0 e^{-t/RC} \Rightarrow dU = VI_0 e^{-t/RC} dt$

$$\Rightarrow U = VI_0 \int_0^\infty e^{-t/RC} dt$$

$$\Rightarrow U = VI_0 \left(-RCe^{-t/RC}\right|_0^\infty \Rightarrow$$

$$\Rightarrow U = RC \cdot VI_0, \quad I_0 = \frac{V}{R}$$

$$\Rightarrow U = RCV\frac{V}{R} \Rightarrow \boxed{U = CV^2}$$

so the battery delivers: $U_b = CV^2$

But energy stored in the cap is: $U_c = \frac{1}{2}CV^2$

$$\boxed{\frac{U_c}{U_b} = \frac{1}{2}}$$

the rest is dissapated through the resistor.