Name: ________________________ **Problems Solved __/ 7**

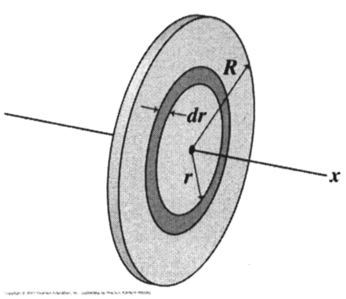
The figure below shows a thin, uniformly charged disk of radius R. Imagine the disk divided into rings of varying radii r, as suggested in the figure.

(a) Show that the area of such a ring is very nearly $2\pi r\, dr$.

(b) If the surface charge density on the disk is σ C/m^2, use the result of part (a) to write an expression for the charge dq on an infinitesimal ring.

(c) Use the result of part (b) along with the electric field of a charged ring a distance x along the axis: $E = \frac{kQx}{(x^2+a^2)^{3/2}}$ to write the infinitesimal electric field dE of this ring at a point on the disk axis, taken to be the positive x axis.

(d) Integrate over all such rings (from $r = 0$ to $r = R$) to show that the net electric field on the disk axis has magnitude $E = 2\pi k\sigma\left(1 - \frac{x}{\sqrt{x^2+R^2}}\right)$



a) $C = 2\pi r$, width is dr

so, $\boxed{dA = 2\pi r\, dr}$

b) $dq = \sigma\, dA = \underline{\sigma 2\pi r\, dr}$

c+d) $$\vec{E} = \int d\vec{E} = \int_0^R \frac{kx\, dq}{(x^2+a^2)^{3/2}}\hat{\imath} = \int_0^R \frac{kx\, 2\pi\sigma r}{(x^2+r^2)^{3/2}}\, dr\, \hat{\imath}$$

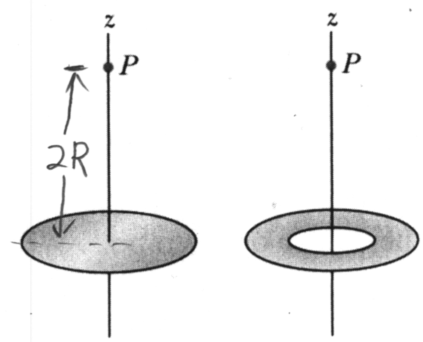
Let $u = x^2 + r^2 \Rightarrow du = 2r\, dr$, $r = (u^2 - x^2)^{1/2}$

$$\Rightarrow \vec{E} = \int_{x^2}^{x^2+R^2} \frac{kx\sigma\pi}{u^{3/2}}\, du\, \hat{\imath} = kx\sigma\pi\left(-2u^{-1/2}\right)\Big|_{x^2}^{x^2+R^2}\hat{\imath} =$$

$$= -2kx\sigma\pi\left(\frac{1}{(x^2+R^2)^{1/2}} - \frac{1}{x}\right)\hat{\imath}$$

$$\Rightarrow \boxed{\vec{E} = 2k\sigma\pi\left[1 - \frac{x}{(x^2+R^2)^{1/2}}\right]\hat{\imath}}$$

Suppose that you design an apparatus in which a uniformly charged disk of radius R is to produce an electric field. The filed magnitude is most important along the central perpendicular axis of the disk, at a point P a distance $2R$ from the disk. Cost analysis suggests that you use a ring with the same outer radius but an inner radius of R/2. Assume that the ring will have the same surface charge density as the original disk. By what percentage will the field be decreased with the new disk?



(a) (b)

Start with the integral for a solid disk and adjust the limits.

$$\vec{E}_b = \int_{\frac{1}{2}R}^{R} \frac{2\pi k z \sigma r}{(z^2+r^2)^{3/2}}\,dr\,\hat{k}, \qquad u = z^2+r^2,\ du = 2r\,dr$$

$$= \int_{z^2+\frac{1}{4}R^2}^{z^2+R^2} \frac{\pi k z \sigma}{u^{3/2}}\,du = \pi k z \sigma \left(-2u^{-1/2}\right)\Big|_{z^2+\frac{1}{4}R^2}^{z^2+R^2}$$

$$= 2\pi k \sigma \left[\frac{z}{(z^2+\frac{1}{4}R^2)^{1/2}} - \frac{z}{(z^2+R^2)^{1/2}}\right]\hat{k}, \qquad z = 2R$$

$$= 2\pi k \sigma \left[\frac{2R}{(4R^2+\frac{1}{4}R^2)^{1/2}} - \frac{2R}{(4R^2+R^2)^{1/2}}\right]\hat{k}$$

$$\boxed{\vec{E}_b = 2\pi k \sigma \left[\frac{4}{\sqrt{17}} - \frac{2}{\sqrt{5}}\right]\hat{k}}$$

continued ↓

Electrostatics Set 4, P2 continued

$$\vec{E}_a = 2k\pi\sigma\left[1 - \frac{z}{(z^2+R^2)^{1/2}}\right]\hat{k}, \quad z = 2R$$

$$= 2\pi k\sigma\left[1 - \frac{2}{\sqrt{5}}\right]\hat{k}$$

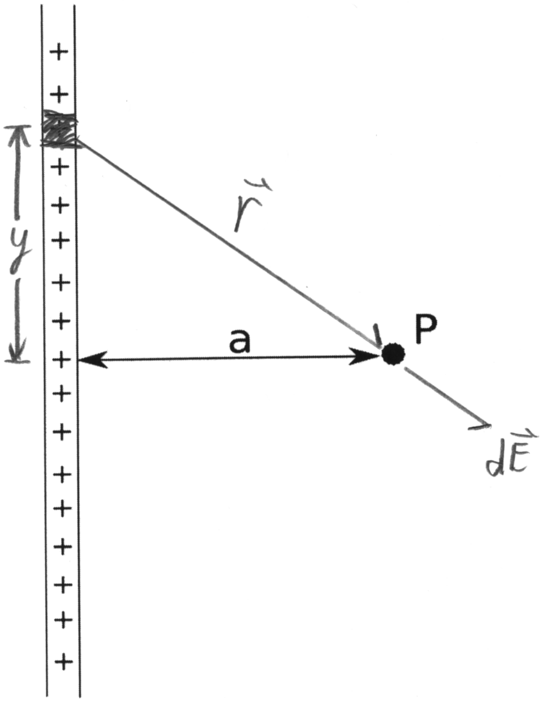
% loss: $$\frac{|\vec{E}_b|}{|\vec{E}_a|} = \frac{\left[\frac{4}{\sqrt{17}} - \frac{2}{\sqrt{5}}\right]}{\left[1 - \frac{2}{\sqrt{5}}\right]} \cdot \frac{\cancel{2\pi k\sigma}}{\cancel{2\pi k\sigma}}$$

$$\boxed{\frac{|\vec{E}_b|}{|\vec{E}_a|} = 0.72}$$

So, decreased by 28%

a) Calculate the field a distance a away from an infinitely long wire with linear charge density λ.
b) Show that the field next to a line of charge of length L becomes the field of an infinite line as a approaches zero.

The field of a line of length L and charge Q is: $\vec{E} = \frac{kQ}{a\sqrt{a^2 + L^2/4}}$



$$\vec{E} = \int d\vec{E} = \int \frac{k\,dq}{|\vec{r}|^2}\hat{r}$$

$$\vec{r} = a\hat{i} + y\hat{j}, \quad dq = \lambda\,dy$$

$$\vec{E} = \int_{-\infty}^{\infty} \frac{k\lambda}{(a^2+y^2)^{3/2}}(a\hat{i} + y\hat{j})\,dy$$

$\hat{j}$ component cancels by symmetry

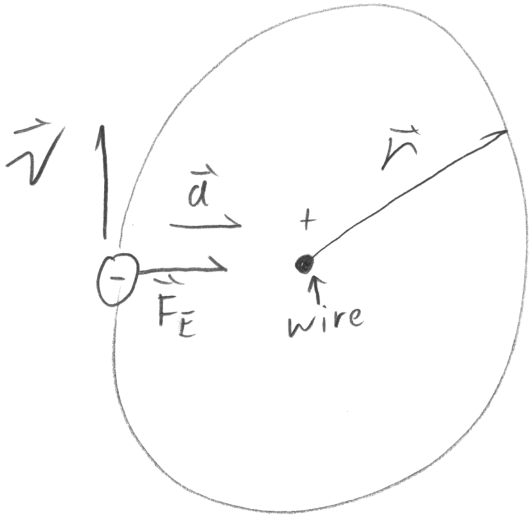
$$\Rightarrow \vec{E} = \int_{-\infty}^{\infty} \frac{k\lambda a}{(a^2+y^2)^{3/2}}dy\,\hat{j} \quad \text{let } y = a\tan\theta \Rightarrow dy = a\sec^2\theta\,d\theta$$

$$\Rightarrow \vec{E} = \int_{-\pi/2}^{\pi/2} \frac{k\lambda a}{a^3 \sec^3\theta}\, a\cancel{\sec^2\theta}\, d\theta\,\hat{i} = \frac{k\lambda}{a}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta\,\hat{i}$$

$$\Rightarrow \boxed{\vec{E} = \frac{2k\lambda}{a}\hat{i}}$$

continued ↓

Imagine that a particle with charge $-q$ and mass m follows a circular orbit around an infinitely long wire (the wire is perpendicular to the orbial plane). Find a formula for the particle's orbital speed in terms of q, m, the wire's charge density λ, and the constant k.



Circular motion

$$\vec{a} = -\frac{v^2}{r}\hat{r} \quad \text{centripetal acceleration}$$

$$\vec{F}_E = -\frac{2k\lambda q}{r}\hat{r} \quad \text{Force due to wire}$$

$$\vec{F} = m\vec{a}$$

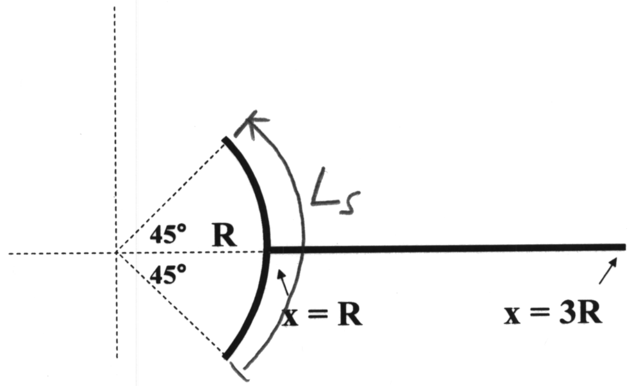
$$-\frac{2k\lambda q}{\cancel{r}}\hat{r} = -\frac{v^2}{\cancel{r}}\hat{r}$$

$$\Rightarrow \boxed{v = (2k\lambda q)^{1/2}} \quad \text{independent of } r\,!$$

A total charge $+Q$ is uniformly distributed over the shape shown in the figure (an arc connected to a straight rod).

(a) Write an expression for the linear charge density λ of the entire object in terms of Q and R. Simplify your expression.

(b) Find the electric field vector at the origin in terms of given quantities and appropriate constants. Show all steps; don't just use previously derived answers. Simplify your expression and write it in terms of k, Q, and R.



Hint: Think of the rod and arc separately, then combine the field vectors from each. (You've seen each of those problems by themselves.)

a) $\lambda = \frac{Q}{L_T}$ where L is the total length of object

$L_T = L_s + L_L$, L_s = length of arc
L_L = length of line

$L_s = \pi/2 \, R$ (because $s = R\theta$)

$L_L = 2R$

$L_T = (\pi/2 + 2)R \Rightarrow \boxed{\lambda = \frac{Q}{R} \cdot \frac{2}{\pi + 4}}$

b) Get $\vec{E}_s$ (field due to arc) first and then $\vec{E}_L$ (field due to the line) and add them together.

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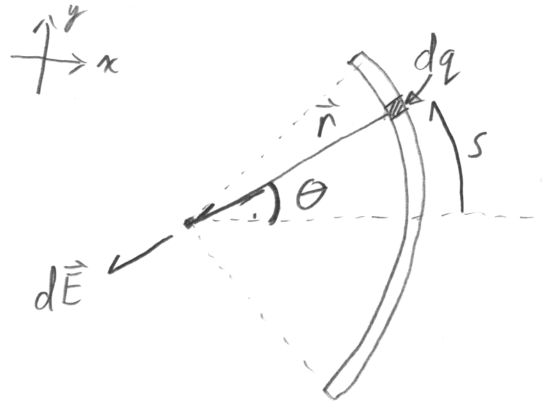
Electrostatics Set 4, P6 - continued



$$\vec{E}_s = \int d\vec{E}$$

$$= \int \frac{k\,dq}{|\vec{r}|^2}\hat{r}$$

$\vec{r} = R(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$ $\qquad dq = \lambda ds = \lambda R\,d\theta$

$|\vec{r}| = R$

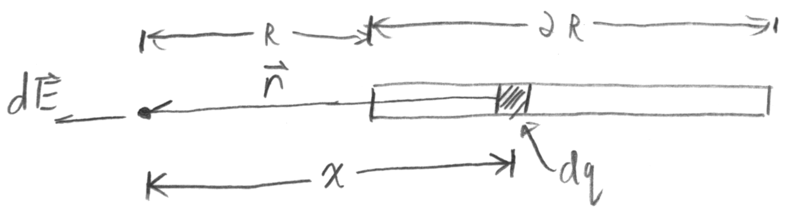
$$\Rightarrow dq = \frac{2Q}{\pi+4}d\theta$$

$$\vec{E}_s = \int_{-\pi/4}^{\pi/4} \frac{k}{R^2}\cdot\frac{2Q}{\pi+4}\left(-\cos\theta\,\hat{\imath} + \overset{0}{\cancel{\sin\theta\,\hat{\jmath}}}\right)d\theta,$$

$\hat{\jmath}$ component cancels by symmetry

$$= \frac{2kQ}{R^2(\pi+4)}\left(-\sin\theta\Big|_{-\pi/4}^{\pi/4}\hat{\imath} = \frac{2kQ}{R^2(\pi+4)}\cdot\frac{2}{\sqrt{2}}\hat{\imath}\right.$$

$$\Rightarrow \boxed{\vec{E}_s = \frac{-2\sqrt{2}\,kQ}{R^2(\pi+4)}\hat{\imath}}$$



$dq = \lambda\,dx$

$$dq = \frac{2Q}{R(\pi+4)}dx$$

$\vec{r} = -x\hat{\imath}$, $\hat{r} = -\hat{\imath}$

continued ↓

$$\vec{E}_L = \int d\vec{E} = \int \frac{k\,dq}{|\vec{r}|^2}\hat{r}$$

$$= -\int_R^{2R} \frac{k}{x^2}\,\frac{2Q}{R(\pi+4)}\,dx\,\hat{\imath}$$

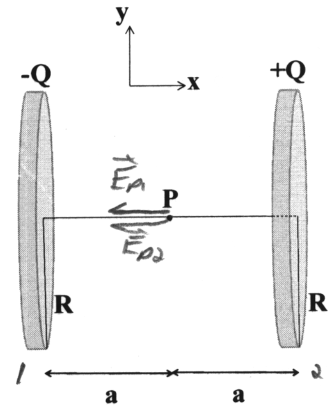
$$= -\frac{2kQ}{R(\pi+4)}\left(-\frac{1}{x}\right|_R^{2R}\hat{\imath} = \frac{2kQ}{R(\pi+4)}\left(\frac{1}{2R}-\frac{1}{R}\right)$$

$$\boxed{\vec{E}_L = -\frac{kQ}{R^2(\pi+4)}}$$

So, putting them together:

$$\boxed{\vec{E}_T = -\frac{kQ}{R^2}\,\frac{(1+2\sqrt{2})}{(\pi+4)}\,\hat{\imath}}$$

Two uniformly charged disks of radius R are parallel to each other and have opposite charges $\pm Q$ and surface charge densities $\pm\sigma$. (a) Calculate the electric field vector at the midpoint of their central axis, a distance a from each disk. Show all work (but you may use symmetry to simplify it, and you may use your previous result for a ~~ring~~ disk without deriving it again). Write your answer in terms of k, a, R, and σ.

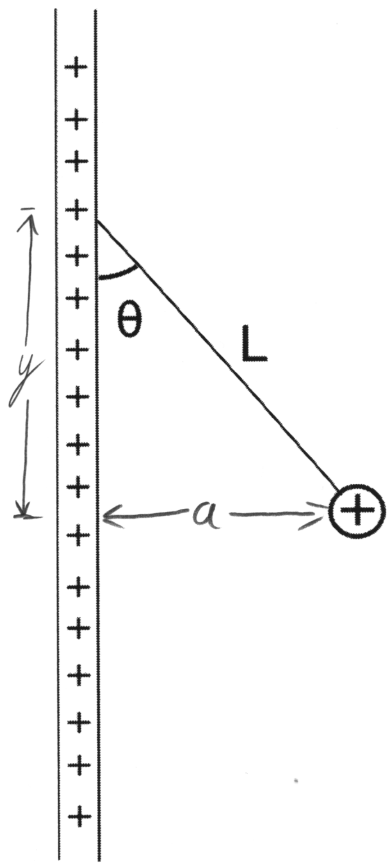


$$\vec{E}_{disk} = 2k\sigma\pi\left[1-\frac{x}{(x^2+R^2)^{1/2}}\right]\hat{x}, \quad \sigma = \frac{Q}{\pi R^2}$$

$$\vec{E}_P = \vec{E}_{P1} + \vec{E}_{P2}$$

$$\boxed{\vec{E}_P = -4k\sigma\pi\left[1-\frac{a}{(a^2+R^2)^{1/2}}\right]\hat{\imath}}$$

A charged ball of mass m and charge q is hanging motionless from a silk thread of length L on a very long charged wire with charge density λ. Assuming that the wire is perpendicular to the floor, find an expression for the distance, a, that the ball will hover from the wire.

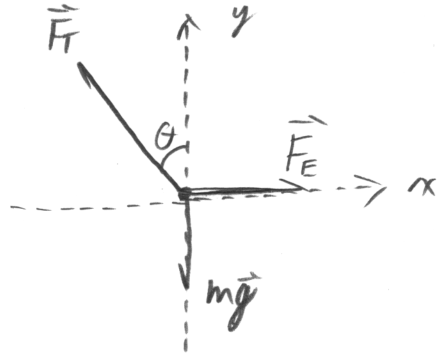


Given

m, q, L, λ

Want

θ



Want $\vec{F}_{net} = 0$ for each axis

$$x: -|\vec{F}_T|\sin\theta + |\vec{F}_E| = 0 \Rightarrow |\vec{F}_E| = |\vec{F}_T|\sin\theta$$

$$y: |\vec{F}_T|\cos\theta - m|\vec{g}| = 0 \Rightarrow m|\vec{g}| = |\vec{F}_T|\cos\theta$$

$$\Rightarrow \tan\theta = \frac{|\vec{F}_E|}{m|\vec{g}|}, \quad |\vec{F}_E| = q|\vec{E}| = \frac{2kq\lambda}{a}$$

$$\Rightarrow \tan\theta = \frac{2kq\lambda}{a\, m|\vec{g}|}, \quad \tan\theta = \frac{a}{y} = \frac{a}{(L^2-a^2)^{1/2}}$$

$$\Rightarrow \frac{a}{(L^2-a^2)^{1/2}} = \frac{2kq\lambda}{a\, m|\vec{g}|} \Rightarrow a^2 = (L^2-a^2)^{1/2}\boxed{\frac{2kq\lambda}{m|\vec{g}|}}$$

Let's call this bundle of constants $\sqrt{b}$ for now

Now, square both sides: $a^4 = (L^2-a^2)\, b$

$\Rightarrow a^4 + ba^2 - bL^2 = 0$; which is quadratic in a^2

$$\Rightarrow \boxed{a^2 = \frac{-b \pm (b^2 + 4bL^2)^{1/2}}{2}}$$

a bit ugly!